MTH314: Discrete Mathematics for Engineers
Graph Theory: Planarity and Coloring

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Planarity

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Because the last 2 can be (for example) drawn like this:
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Exercise: are these planar?
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![Graphs](image-url)
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A planar graph can be divided into faces as well as edges and vertices.
Planarity

The number of faces of a planar graph is fixed in every planar embedding. How many faces do the graphs in exercise 1 have?
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Consider the following planar graph.

1, 4, 5, 6, 7 are boundary vertices of \( f_1 \). 1, 2, 3, 4 are boundary vertices of \( f_2 \). 1, 2, 3, 4, 5, 6, 7, 8 are all boundary vertices of \( f_3 \). The boundary walk of \( f_1 \) is \([1 \ 4 \ 5 \ 6 \ 7 \ 1]\). Those of \( f_2 \) and \( f_3 \) are \([1 \ 2 \ 3 \ 4]\) and \([1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 7 \ 1]\) respectively.
Planarity

A planar embedding of a graph partitions the plane into faces. That’s why the outside of a graph is a face too. A planar graph has one face if and only if it is a tree.

The boundary of a face are the edges and vertices incident to the face. The boundary walk of a face is the closed walk of the incident vertices. It can go over the same edge twice.

The degree of a face is the length of its boundary walk. The length is measured in edges covered.

Two faces are adjacent if they share an edge.
Planarity

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What is the sum of the degrees of all faces?

Every edge contributes one to the degree of two faces, or two to the degree of a single face. So just like the sum of degrees of the vertices, the sum of degrees of the faces is $2e$, where $e$ is the number of edges.
Planarity

Every planar embedding has the same number of faces $f$. (And obviously the same number of vertices and edges.) Some properties may depend on the embedding though. These are two different planar embeddings of the same graph:

One of them has a face of degree 5, and the other one doesn’t.
Planarity: Euler’s Formula

Theorem

In a planar connected graph with $v$ vertices, $e$ edges, and $f$ faces we have:

$$v - e + f = 2$$
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If it is not a tree, there exists an edge that is incident to two different faces. If we remove it, both the number of edges and the number of faces go down by one. So \( v - e + f \) stays the same. As long as the graph is not a tree, remove such edges one by one. Eventually we are left with a tree. □
Planes and Spheres

**Theorem:** A graph is planar if and only if it can be embedded on the surface of a sphere so that no two edges cross.

**Example:** The 5-clique can be embedded on the surface of the torus so that no two edges cross. Can you embed the 5-clique on the plane?
**Definition:** In geometry, a **Platonic solid** is a convex polyhedron that is **regular**.

This means that the faces of a Platonic solid are congruent regular polygons, with the same number of faces meeting at each vertex.

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- Cube
- Octahedron
- Dodecahedron
- Icosahedron
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A Platonic graph is a d-regular and c-face-regular planar graph. Because of the sphere condition, for every Platonic solid we must have a Platonic graph and vice versa.
A Platonic graph is a d-regular and c-face-regular planar graph. (Every vertex has d edges, every face has c edges incident.)

All planar graphs obey Euler’s formula $v - e + f = 2$. Also, we have:

$$2e = dv,$$

$$2e = cf.$$
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Since \( e, c, d \) are all positive integers, we must also have \( 2c + 2d - dc > 0 \). We can eventually find that the solutions for \((d, c)\) are \((3, 3), (3, 4), (4, 3), (3, 5), (5, 3)\). Hence there are exactly 5 Platonic solids.
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Ex: check that these values for $d, c$ give the right values for $v, f$. 

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So if we find that a graph is too dense, we can immediately conclude it is not planar. For example, if $v = 5$, in order for $6 + e > 15$ we need to have 10 edges. The 5-clique is the only graph on 5 vertices that is too dense to be planar. This doesn't work the other way. Just because graph is not dense, it doesn't mean it’s planar. $3v \geq 6 + e$ is a necessary but not sufficient condition.
Is it planar?
The Petersen Graph

Theorem: The Petersen graph is not planar.

Proof: Suppose for contradiction that we can find a planar embedding of this graph. Then we can also find a planar embedding of the 5-clique. but by density argument, we know that’s impossible. We conclude that the Petersen graph is not planar.

□
We can make this argument because we see that the structure of the 5-clique is somehow included in the Petersen graph. More precisely, if we merge some vertices by contracting edges, we get the 5 clique. If you think of a graph as a data structure, we only lose structure by combining two data entries, don’t add any.
Here are graph operations that reduce the structure of a graph:

**Vertex deletion:** remove any vertex, along with all incident edges.

**Edge deletion:** remove any edge.

**Edge contraction:** remove an edge and merge together its two incident vertices. Also remove possible parallel edges.

**Example:**

\[ G: \]

\[ H: \]

\( H \) is a minor of \( G \).

If a graph \( H \) can be obtained from a graph \( G \) by a sequence of these operations, then \( H \) is a minor of \( G \).

The 5-clique is a minor or the Petersen graph.

If a graph \( G \) has a minor \( H \) that is not planar, \( G \) is not planar.
Kuratowski’s Theorem

Theorem (Kuratowski’s Theorem)

A graph is non-planar if and only if it has either of these two graphs (or both) as a minor:
Graph colouring

Informally, a $k$-colouring of a graph $G$ is an assignment of $k$ colours to the vertices of $G$ so that no two adjacent vertices have the same colour.

More formally, it’s a function $f : V \rightarrow \{1, 2, \ldots, k\}$ so that for any $v, w \in V$,

$$\{v, w\} \in E \Rightarrow f(v) \neq f(w),$$

that is, if two vertices are adjacent $f(v) \neq f(w)$. 
Graph colouring

If we can find a coloring of a graph with \( k \) colours, that graph is \( k \)-colourable. We’ve seen 2-colourable graphs before, what were they called then?
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A graph is $k$-colourable if and only if there is a partition of the vertices into at most $k$ independent sets.
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**Proof:** \( \Rightarrow \) Every colour defines a subset of vertices which is independent.

\( \Leftarrow \) An independent set can be coloured with one colour. \( \square \)
Colouring cliques

A $k$-clique is $k$-colourable but not $(k - 1)$-colourable.
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Proof:

1. (A $k$–clique is $k$-colorable.) In a $k$-coloring, every vertex gets a different colour, so every graph with $k$ vertices is $k$ colourable.

2. (A $k$–clique is not $(k - 1)$-colorable.) Suppose now (for the sake of contradiction) that there exists a colouring of the $k$-clique with $(k1)$ colours. By the pigeonhole principle, there are two vertices with the same colour. But the graph is a clique, so these vertices must be adjacent! □
Theorem

*Every planar graph is 6-colourable.*
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Proof: the proof will proceed in two parts.

1. Every planar graph contains a vertex with degree at most 5.
2. We will use this fact to prove the theorem by induction on the number of vertices.
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1. Suppose that the graph is planar. Then the number of vertices $n$ and the number of edges $e$ obey the inequality: $3n - 6 \geq e$. If no vertex has degree at most 5, then the sum of degrees is $2e \geq 6n$. But then $e \geq 3n$, so the graph can’t be planar.
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2. Base case: let $n = 1$. Clearly, the graph is 6-colourable. (In fact, for every $n \leq 6$, every graph is 6-colourable.)
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2. Base case: let $n = 1$. Clearly, the graph is 6-colourable. (In fact, for every $n \leq 6$, every graph is 6-colourable.)

Inductive hypothesis: suppose that for some $n$, every planar graph on $n$ vertices is 6-colourable.
So take any planar graph on $n + 1$ vertices. This graph has at least one vertex $v$ of degree at most 5. If we removed that vertex, the remaining graph would be a planar graph on $n$ vertices, and therefore 6-colourable by the inductive hypothesis.
So take any planar graph on \( n + 1 \) vertices. This graph has at least one vertex \( v \) of degree at most 5. If we removed that vertex, the remaining graph would be a planar graph on \( n \) vertices, and therefore 6-colourable by the inductive hypothesis.

So take any 6-colouring of the vertices other than \( v \). \( v \) has at most 5 neighbors, and so at most 5 colours were used to colour them. Assign any remaining colour to \( v \). This is a 6-colouring of the original graph.
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In fact every planar graph is 4-colourable. This is called the 4 Colour Theorem and extensive computer power was necessary to prove it. There’s a philosophical debate among mathematicians about whether that constitutes a valid proof.
Examinable material ends here.
Real-life networks, like the social networks that represent Facebook or Twitter have these properties:
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- Large scale
- Evolving over time (new users join, new edges appear)
- Degree distribution (how many people are there with a certain number of friends/followers) decreases exponentially... what do you expect to be the difference between Facebook and Twitter?
- They’re highly clustered- people who have friends in common are more likely to know each other. Communities emerge.
- You can get from one user to another in a small number of “hops.”
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There are 4 vertices of degree 6 here, and 3 vertices of degree 4. So most vertices have a higher degree than the average of their neighbors.
PageRank

PageRank is the algorithm that decides what appears first in a Google search, and it’s a graph algorithm.

Think of the information on the internet as a graph, where webpages/content accessible through the internet are nodes, and there is a directed link going from page A to page B if page A includes a hyperlink to page B. To get the “rank” of the page represents how much time on average a token following the links randomly would spend on the page.
Random walk on a graph: follow every available link with the same probability.

\[ \frac{1}{2} \quad \frac{1}{2} \]

0 \quad 1 \quad 2
Random walk on a graph: follow every available link with the same probability.

This token spends twice as much time in the middle as either of the ends. This amount of time is called a stationary distribution of a random walk. (Sometimes called a Markov chain.) This stationary distribution is the rank in PageRank.
Random walk on a graph: follow every available link with the same probability.

\[
\begin{array}{c}
0 \quad 1 \quad 2 \\
\end{array}
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This token spends twice as much time in the middle as either of the ends. This amount of time is called a *stationary distribution* of a random walk. (Sometimes called a Markov chain.) This stationary distribution is the rank in PageRank. (There are some more details to it of course... for example Google’s walker randomly reappears in a different part of the web sometimes to avoid sinking into loops and dead ends.)

Fun fact: the “Page” in PageRank doesn’t really stand for webpages. It’s for Larry Page, one of the inventors.