MTH314: Discrete Mathematics for Engineers

Graph Theory: Planarity and Coloring

Dr Ewa Infeld

Ryerson University

Planarity

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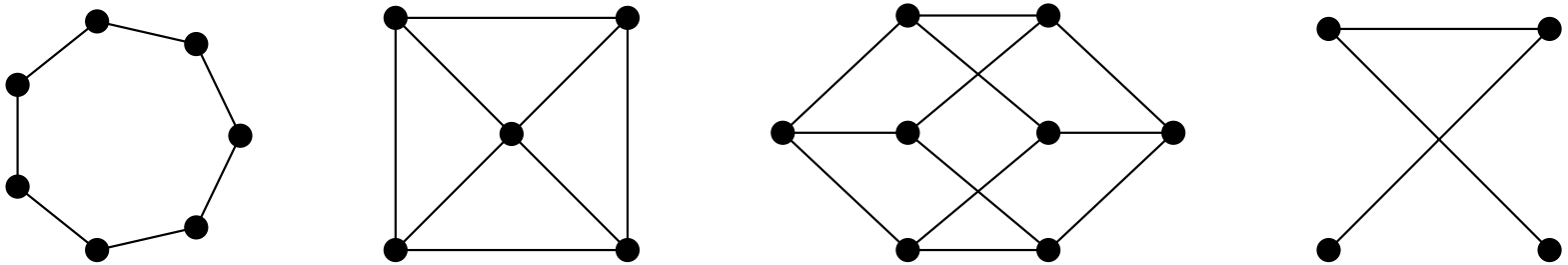
These graphs are all planar:



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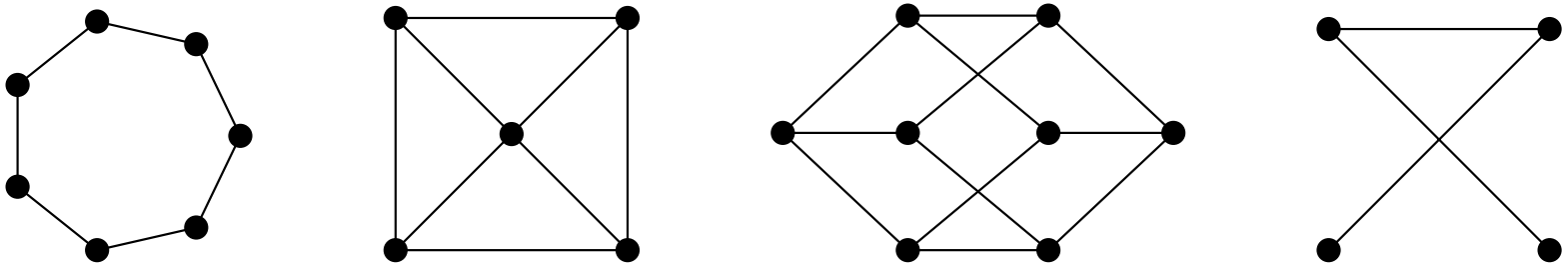
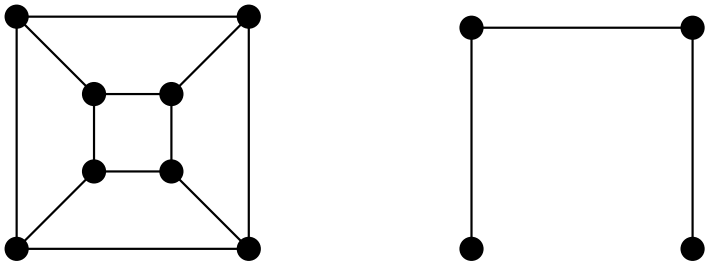


Because the last 2 can be (for example) drawn like this:



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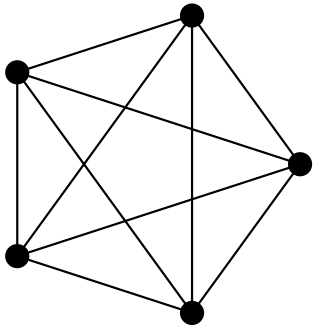

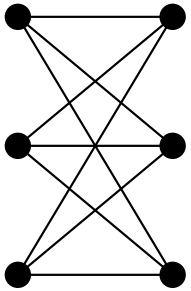

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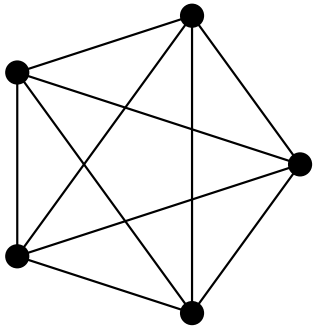
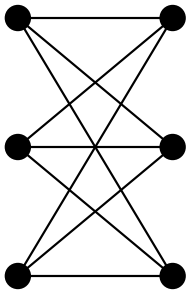
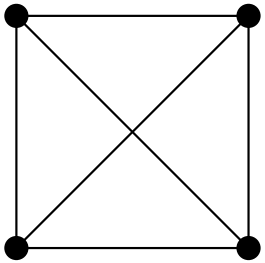
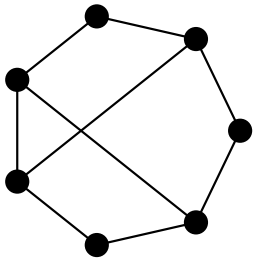
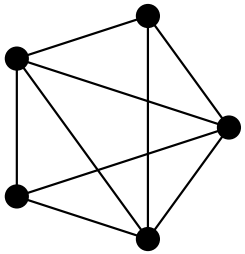


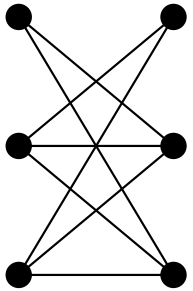


Exercise: are these planar?









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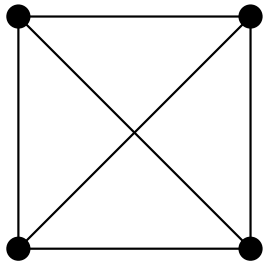

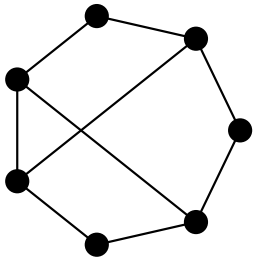

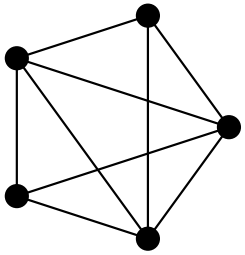

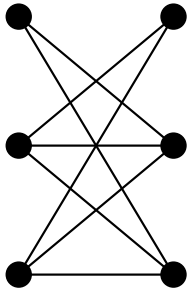

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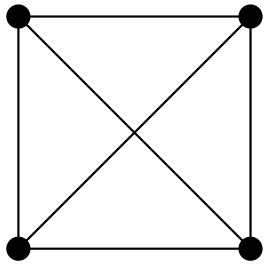

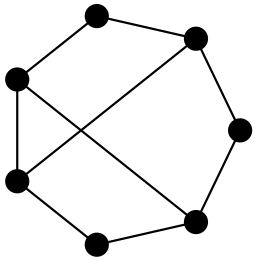

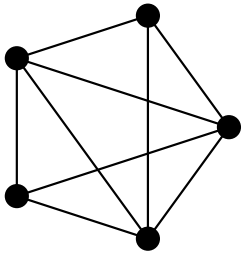

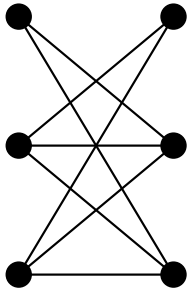

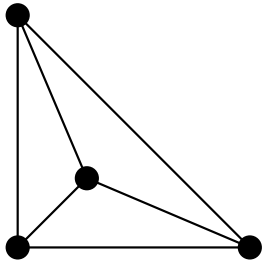

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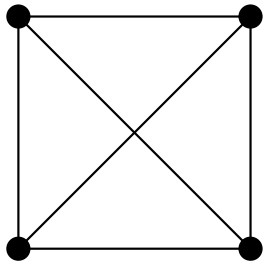

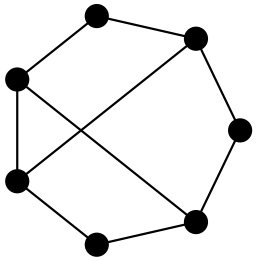

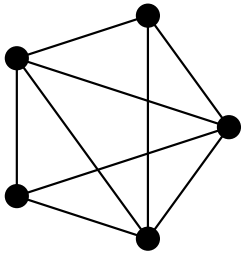

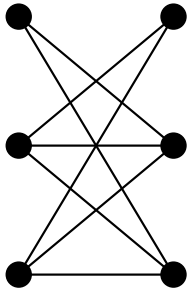

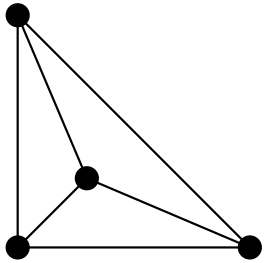

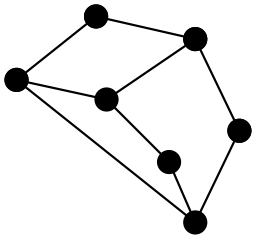

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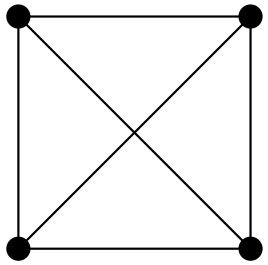
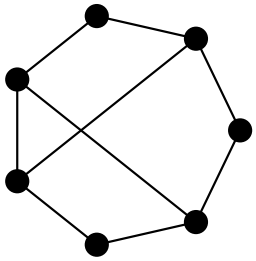
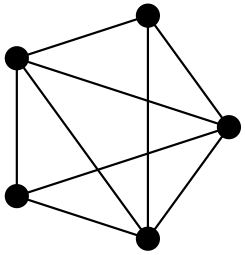
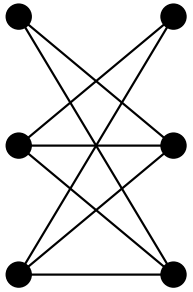
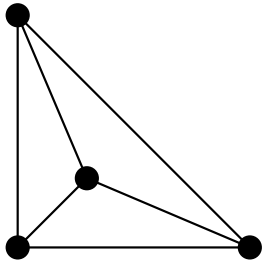
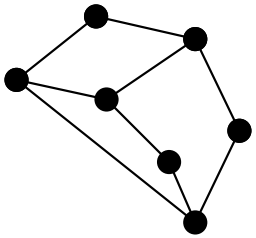
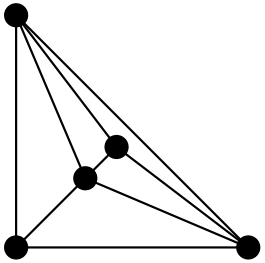

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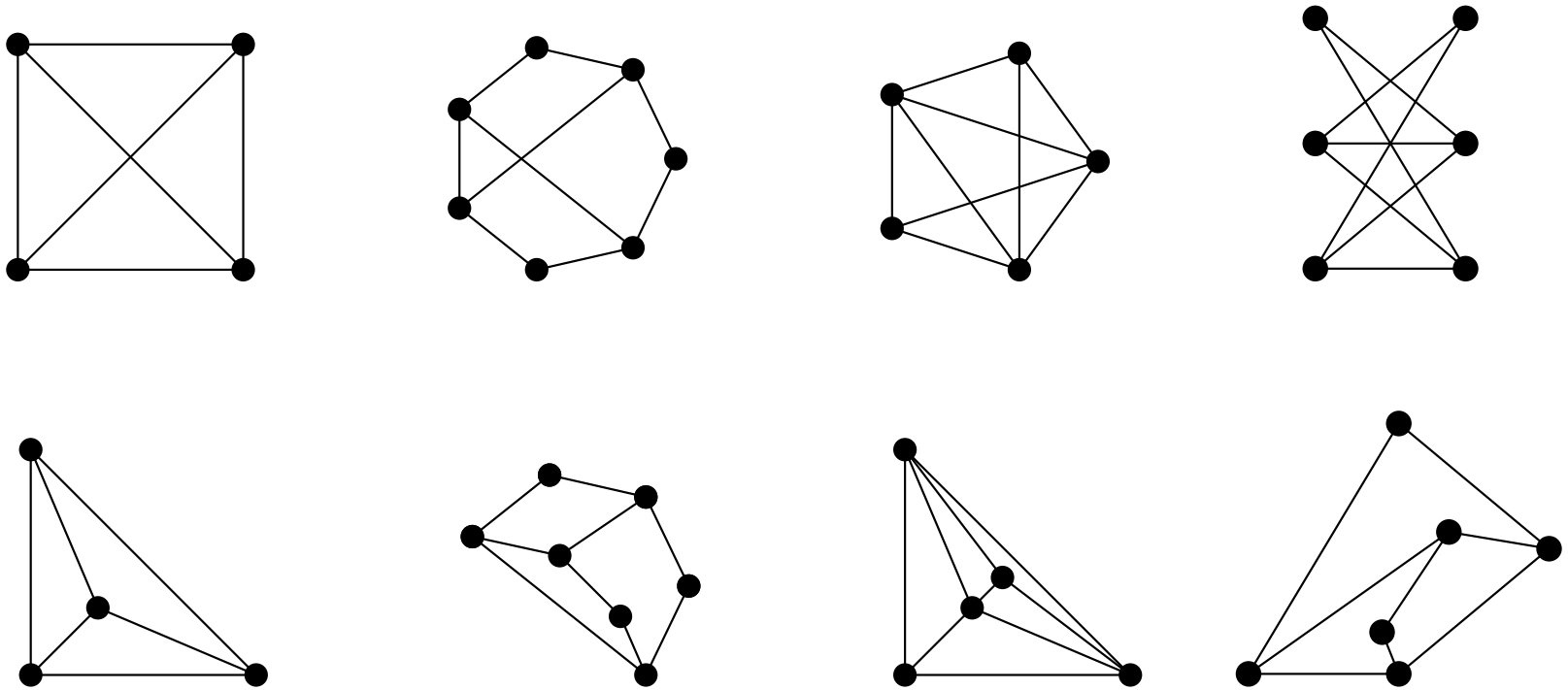

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Definition (Embedding)

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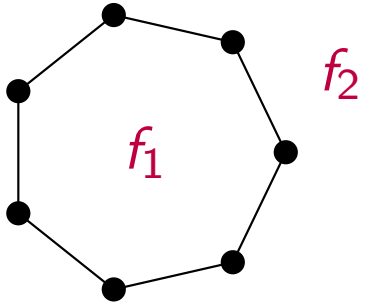
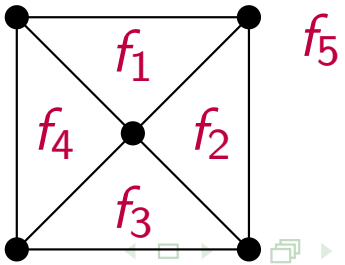
A planar graph can be divided into faces as well as edges and vertices.





Planarity

The number of faces of a planar graph is fixed in every planar embedding. How many faces do the graphs in exercise 1 have?

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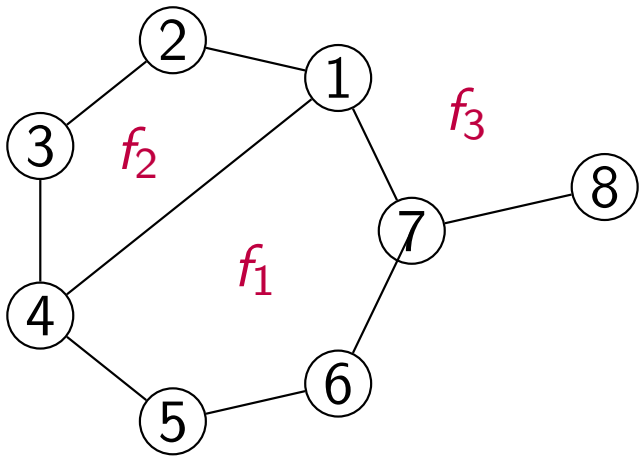
Consider the following planar graph.



1,4,5,6,7 are boundary vertices of f_1. 1,2,3,4 are boundary vertices of f_2. 1,2,3,4,5,6,7,8 are all boundary vertices of f_3. The boundary walk of f_1 is [1 4 5 6 7 1]. Those of f_2 and f_3 are [1 2 3 4] and [1 2 3 4 5 6 7 8 7 1] respectively.

Planarity

A planar embedding of a graph partitions the plane into faces.

That's why the outside of a graph is a face too.

A planar graph has one face if and only if it is a tree.

The boundary of a face are the edges and vertices incident to the face. The boundary walk of a face is the closed walk of the incident vertices.

It can go over the same edge twice.

The degree of a face is the length of its boundary walk.

The length is measured in edges covered.

Two faces are adjacent if they share an edge.

Planarity

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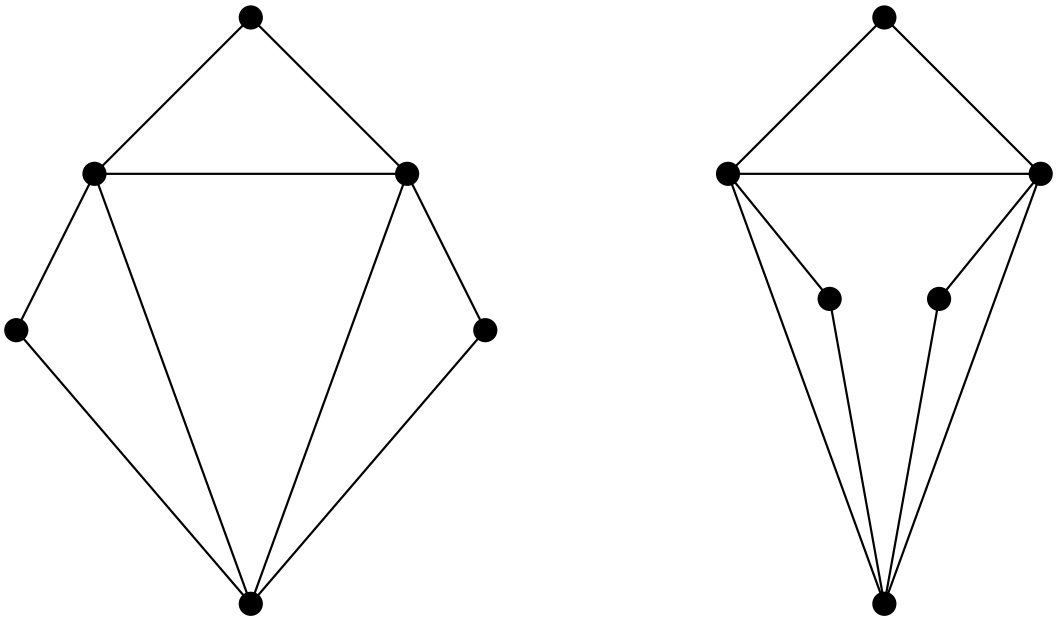
What is the sum of the degrees of all faces?

Every edge contributes one to the degree of teo faces, or two to the degree of a single face. So just like the sum of degrees of the vertices, the sum of degrees of the faces is $2e$, where e is the number of edges.

Planarity

Every planar embedding has the same number of faces f. (And obviously the same number of vertices and edges.)
Some properties may depend on the embedding though. These are two different planar embeddings of the same graph:



One of them has a face of degree 5, and the other one doesn't.

Planarity: Euler's Formula

Theorem

In a planar connected graph with v vertices, e edges, and f faces we have:

$$v - e + f = 2$$

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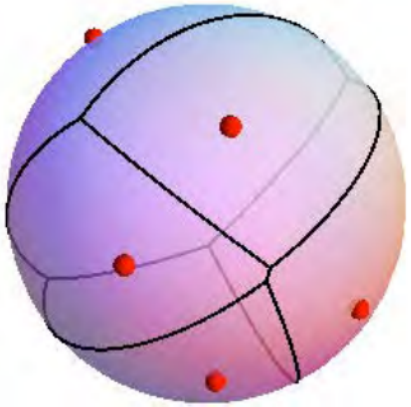
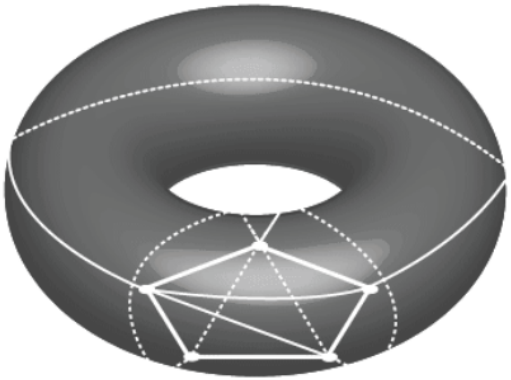
Proof: If the graph is a tree, if has only one face and $e = v - 1$. So the formula holds.

If it is not a tree, there exists an edge that is incident to two different faces. If we remove it, both the number of edges and the number of faces go down by one. So $v - e + f$ stays the same. As long as the graph is not a tree, remove such edges one by one. Eventually we are left with a tree. $\square$

Planes and Spheres

Theorem: A graph is planar if and only if it can be embedded on the surface of a sphere so that no two edges cross.





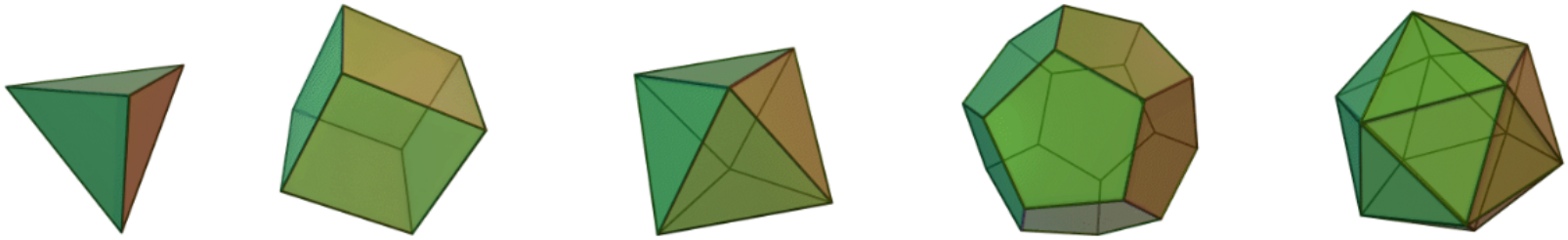
Example: The 5-clique can be embedded on the surface of the torus so that no two edges cross. Can you embed the 5-clique on the plane?

Platonic solids

Definition: In geometry, a **Platonic solid** is a convex polyhedron that is **regular**.

This means that the faces of a Platonic solid are congruent regular polygons, with the same number of faces meeting at each vertex.

Tetrahedron Cube Octahedron Dodecahedron Icosahedron



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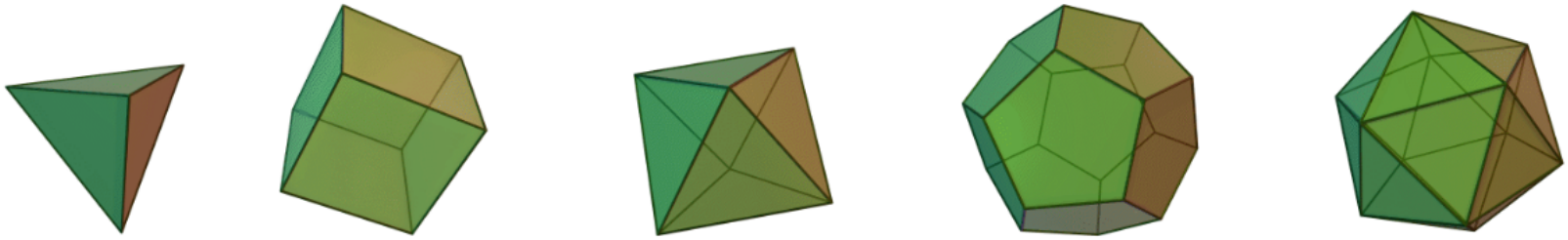

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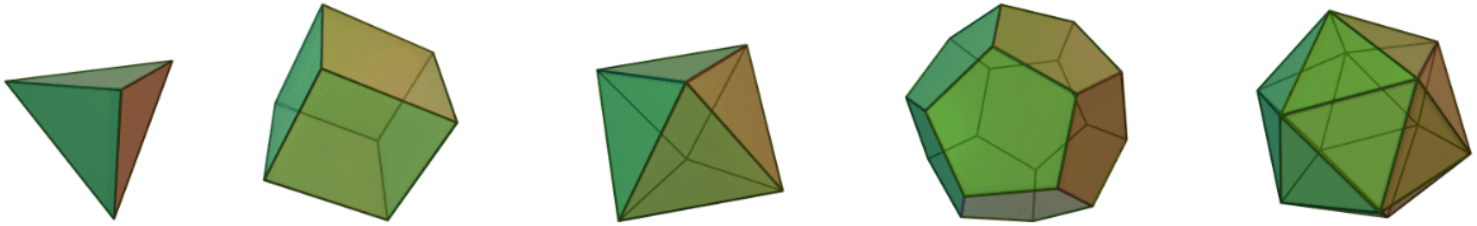
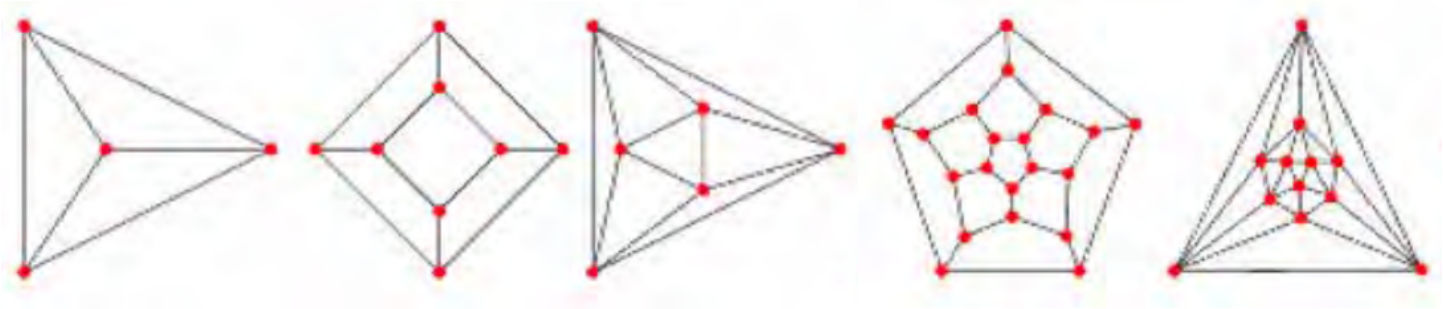


A Platonic graph is a d-regular and c-face-regular planar graph.



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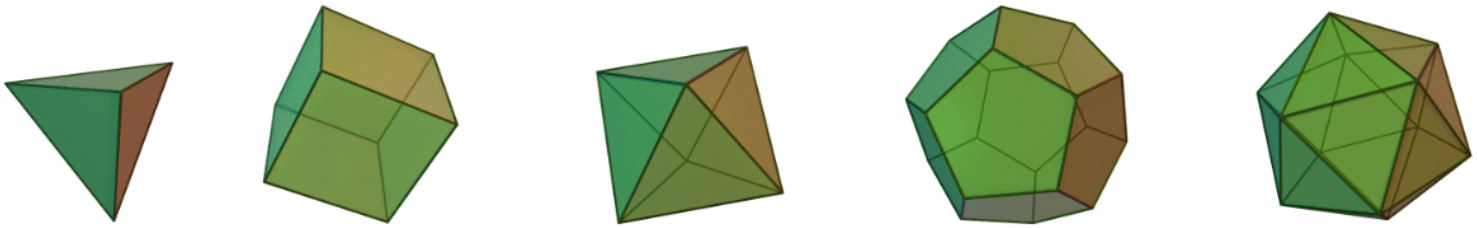

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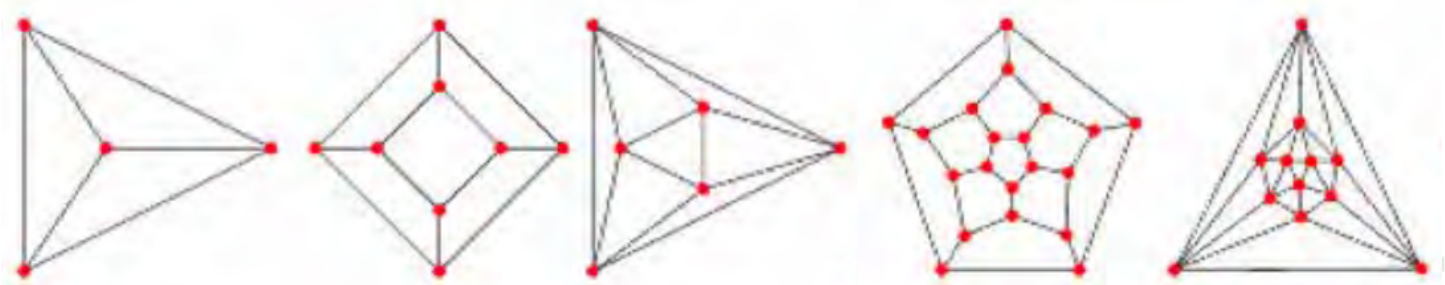

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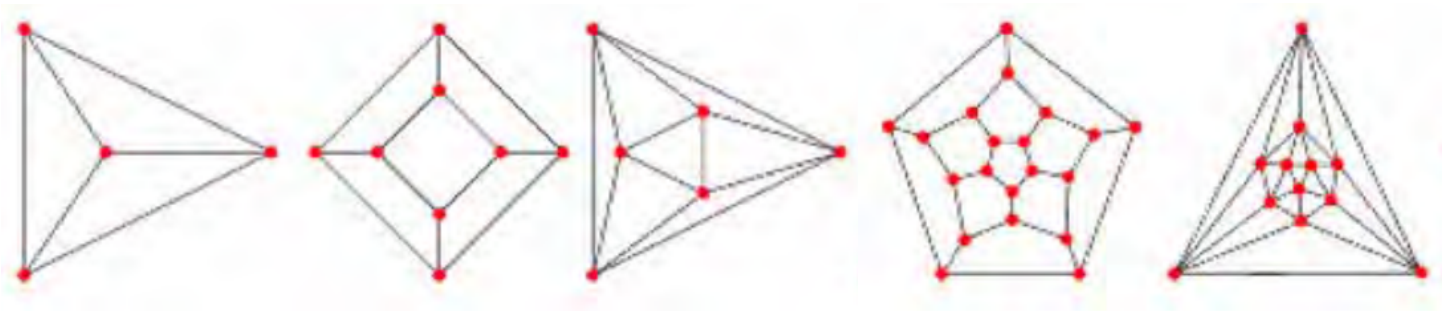

All planar graphs obey Euler's formula $v - e + f = 2$. Also, we have:

$$2e = dv,$$

$$2e = cf.$$

Platonic solids

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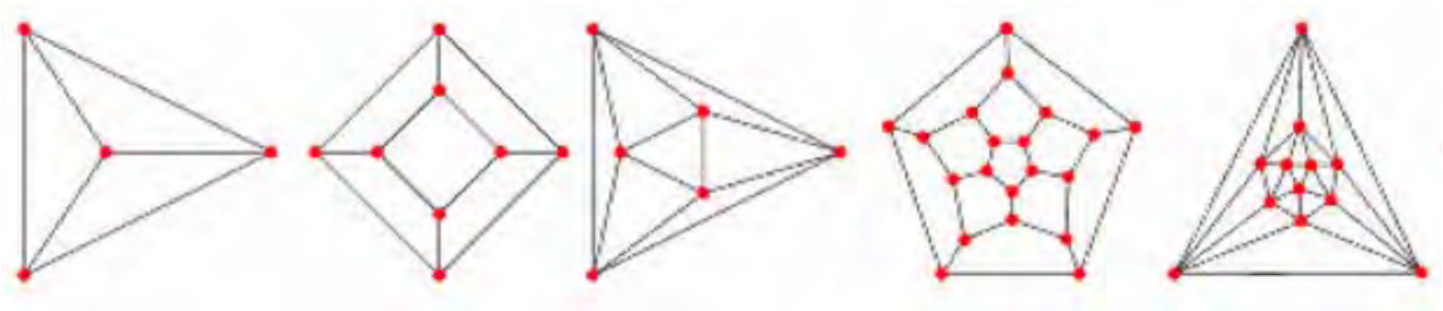

All planar graphs obey Euler's formula $v - e + f = 2$. Also, we have:

$$2e = dv,$$

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So:

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Ex: check that these values for d,c give the right values for v, f.

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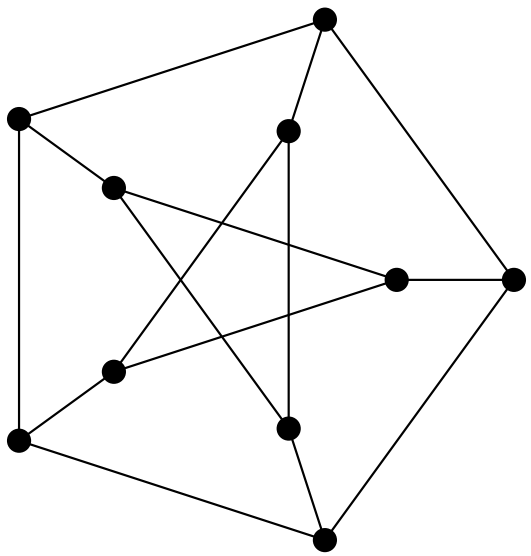
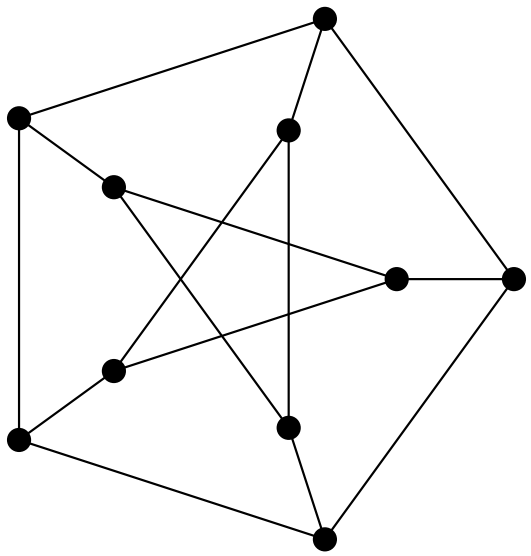
The Petersen Graph



Is it planar?

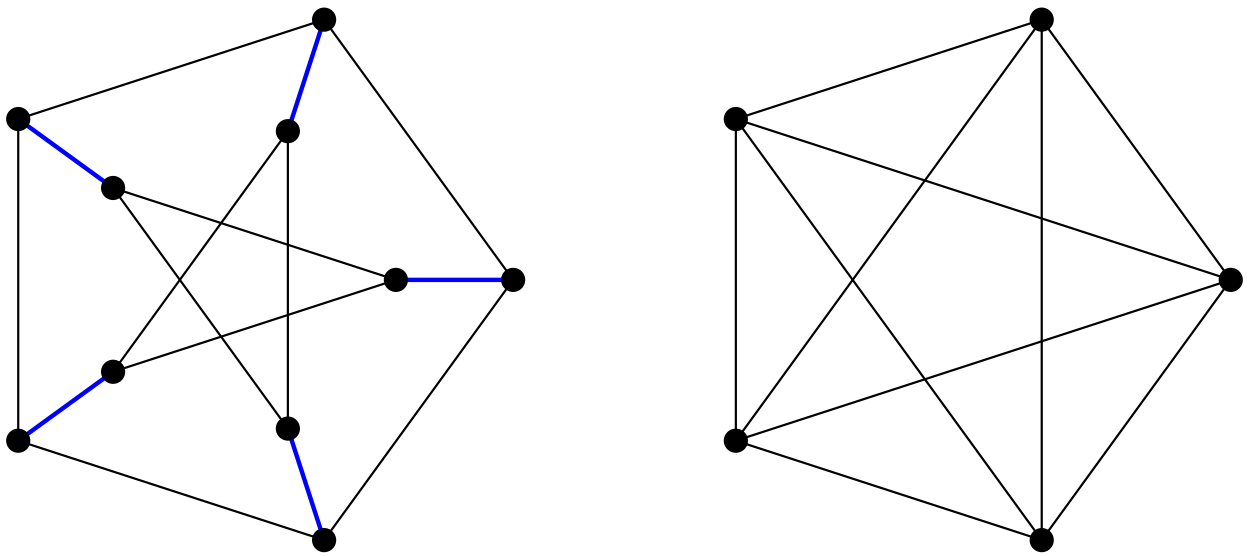
The Petersen Graph



Theorem: The Petersen graph is not planar.

Proof: Suppose for contradiction that we can find a planar embedding of this graph. Then we can also find a planar embedding of the 5-clique. but by density argument, we know that's impossible. We conclude that the Petersen graph is not planar. □

The Petersen Graph



We can make this argument because we see that the structure of the 5-clique is somehow included in the Petersen graph. More precisely, if we merge some vertices by contracting edges, we get the 5 clique. If you think of a graph as a data structure, we only lose structure by combining two data entries, don't add any.

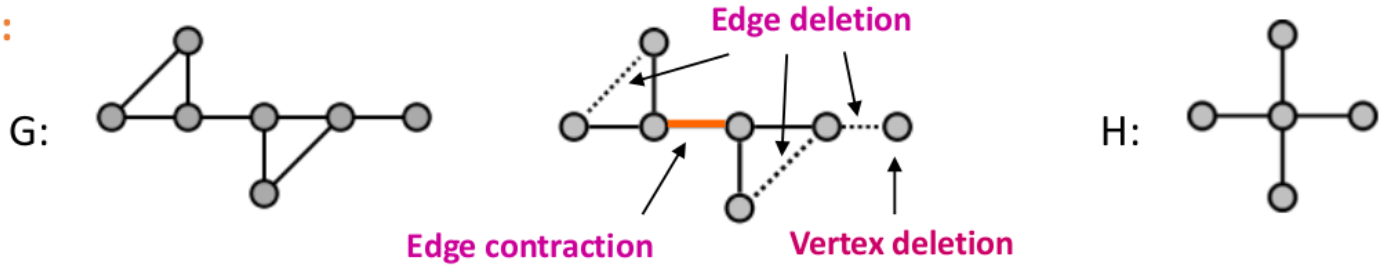
Here are graph operations that reduce the structure of a graph:

Vertex deletion: remove any vertex, along with all incident edges.

Edge deletion: remove any edge.

Edge contraction: remove an edge and merge together its two incident vertices. Also remove possible parallel edges.

Example:



H is a minor of G.

If a graph H can be obtained from a graph G by a sequence of these operations, then H is a minor of G.

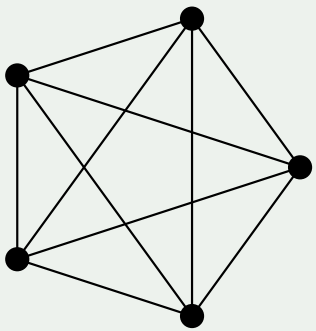
The 5-clique is a minor or the Petersen graph.

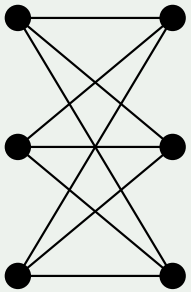
If a graph G has a minor H that is not planar, G is not planar.

Kuratowski's Theorem

Theorem (Kuratowski's Theorem)

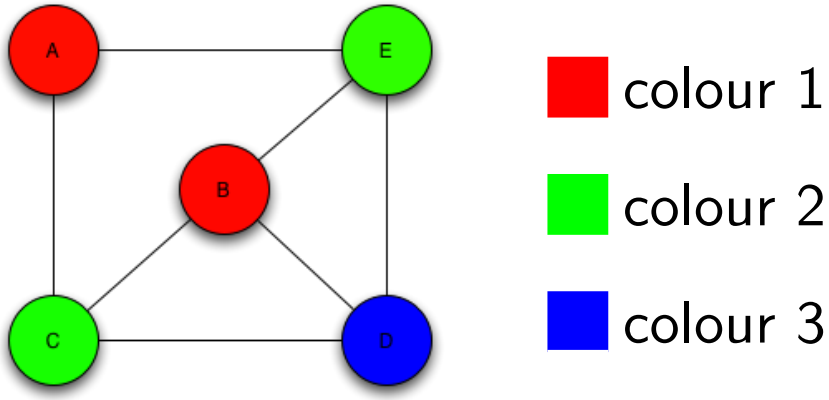
A graph is non-planar if and only if it has either of these two graphs (or both) as a minor:





Graph colouring

Informally, a k-colouring of a graph G is an assignment of k colours to the vertices of G so that no two adjacent vertices have the same colour.



More formally, it's a function $f : V \to \{1, 2, \ldots, k\}$ so that for any $v, w \in V$,

$$[\{v, w\} \in E] \Rightarrow f(v) \neq f(w),$$

that is, if two vertices are adjacent $f(v) \neq f(w)$.

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If we can find a coloring of a graph with k colours, that graph is k-colourable. We've seen 2-colourable graphs before, what were they called then?

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Proof: $\Rightarrow$ Every colour defines a subset of vertices which is independent.
$\Leftarrow$ An independent set can be coloured with one colour. $\square$

Colouring cliques

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Proof:

1. (A $k-$clique is k-colorable.) In a k-coloring, every vertex gets a different colour, so every graph with k vertices is k colourable.
2. (A $k-$clique is not $(k-1)$-colorable.) Suppose now (for the sake of contradiction) that there exists a colouring of the k-clique with $(k1)$ colours. By the pigeonhole principle, there are two vertices with the same colour. But the graph is a clique, so these vertices must be adjacent!

$\square$

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2. Base case: let $n = 1$. Clearly, the graph is 6-colourable. (In fact, for every $n \leq 6$, every graph is 6-colourable.)
   Inductive hypothesis: suppose that for some n, every planar graph on n vertices is 6-colourable.

So take any planar graph on $n+1$ vertices. This graph has at least one vertex v of degree at most 5. If we removed that vertex, the remaining graph would be a planar graph on n vertices, and therefore 6-colourable by the inductive hupothesis.

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In fact every planar graph is 4-colourable. This is called the 4 Colour Theorem and extensive computer power was necessary to prove it. There's a philosophical debate among mathematicians about whether that constitutes a valid proof.

Examinable material ends here.

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- Large scale
- Evolving over time (new users join, new edges appear)
- Degree distribution (how many people are there with a certain number of friends/followers) decreases exponentially... what do you expect to be the difference between Facebook and Twitter?
- They're highly clustered- people who have freinds in common are more likely to know each other. Communities emerge.
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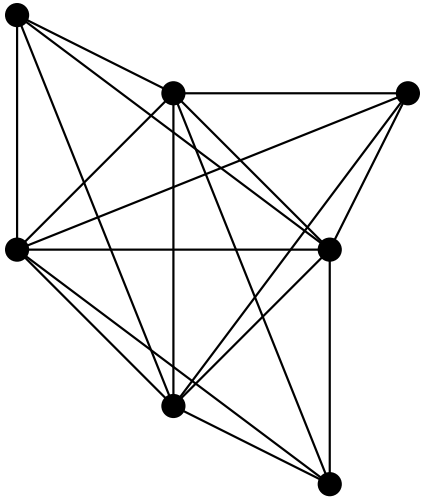
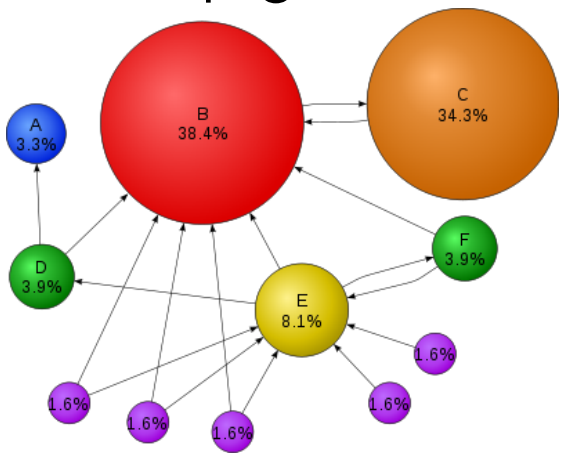


There are 4 vertices of degree 6 here, and 3 vertices of degree 4. So most vertices have a higher degree than the average of their neighbors.

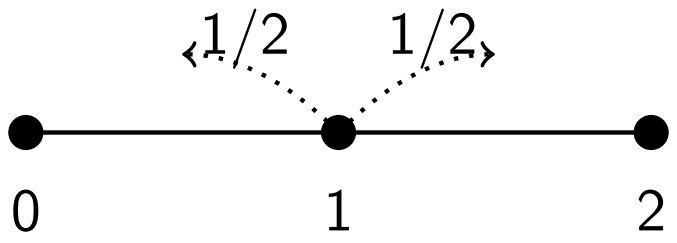
PageRank

PageRank is the algorithm that decides what appears fist in a Google search, and it's a graph algorithm.

Think of the information on the internet as a graph, where webpages/content accessible through the internet are nodes, and there is a directed link going from page A to page B if page A includes a hyperlink to page B. To get the "rank" of the page represents how much time on average a token following the links randomly would spend on the page.



Random walk on a graph: follow every available link with the same probability.



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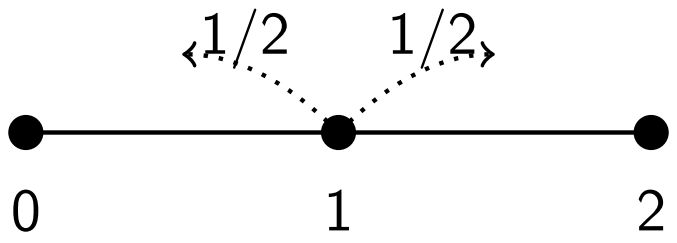

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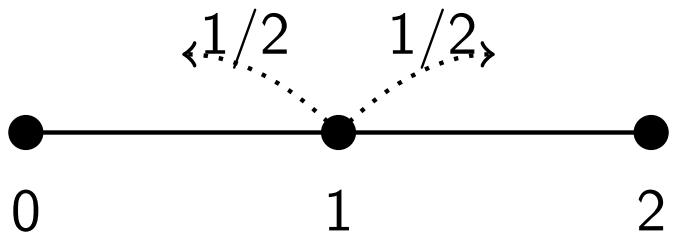

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Fun fact: the "Page" in PageRank doesn't really stand for webpages. It's for Larry Page, one of the inventors.