A few words on Assignment 2

Question 2: D is the set of all students
$M(s)$: "s is a math major."
$C(s)$: "s is a computer science student."
$E(s)$: "s is an engineering student."

"There is an engineering student who is a math major."

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$$\exists s, t \in D,\ (C(s) \wedge E(s)) \wedge (C(t) \wedge \neg E(t))$$

MTH314: Discrete Mathematics for Engineers

Lecture 3: Set Theory and Pigeonhole Principle

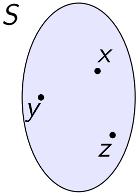
Dr Ewa Infeld

Ryerson Univesity

25 January 2017

Sets

A set is a collection of objects. It is determined by the elements that belong to it.



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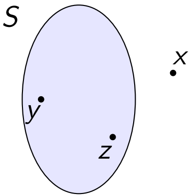

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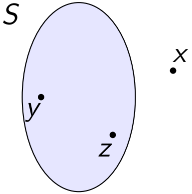
Sets

A set is a collection of objects that obeys Axioms of Set Theory. It is determined by the elements that belong to it.



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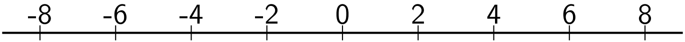
Sets

A set is a collection of objects that obeys Axioms of Set Theory. It is determined by the elements that belong to it.

Axiom of Extensionality: We think of two sets that have the same elements as the same set.

Example:
The set of natural numbers that are multiples of 2, and the set of even numbers are the same set.



Why does it matter? If two different programs compute the same thing, are they the same program?

Set Notation

$x \in S$ reads as "x is an element of S." or "x belongs to S."

$x \notin S$ reads as "x is not an element of S." or "x does not belong to S."

If the set S is a set of breakfast options, and you can pick eggs, oatmeal or fruit, we use this notation:

$$S = \{\text{eggs, oatmeal, fuit}\}$$

Sometimes we see ":=" as in, $S := \{eggs, oatmeal, fuit\}$. This usually happens when you *define* something. You can think of a parallel with programming - the first time you declare S to be something ($S := \{eggs, oatmeal, fuit\}$), vs when you simply state a fact about S, ($S = \{eggs, oatmeal, fuit\}$.) You don't always need to "declare" it in math though.

Set Notation

$T \subseteq S$ reads as "T is a subset of S." It means that every element of T is also an element of S

$$T \subseteq S \leftrightarrow \forall t \in T,\ \exists s \in S,\ t = s$$

$$\forall t \in T,\ t \in S$$

$T \not\subseteq S$ means T is not a subset of S, i.e. some element in T is not an element in S.

$$T \not\subseteq S \leftrightarrow \exists t \in T,\ \forall s \in S,\ t \neq s$$

$$\exists t \in T,\ t \notin S$$

If we write $T = S$, we say T and S are equal, so

$$T = S \leftrightarrow (T \subseteq S) \wedge (S \subseteq T).$$

Set Notation

If we write $T \subset S$, we say T is a subset of S AND it is not equal to S. Notice that, $T \subset S \leftrightarrow (T \subseteq S) \wedge (S \nsubseteq T)$. Then T is a strict or proper subset of S.

If T is a subset of S, then S is a superset of T.

Example:
$T = \{t \in \mathbb{Z} \mid s = 12n + 6 \text{ for some } n \in \mathbb{Z}\}$
$S = \{s \in \mathbb{Z} \mid s = 6m \text{ for some } m \in \mathbb{Z}\}$

Then $T \subseteq S$, but $S \nsubseteq T$.

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Proof outline: If $t = 12n + 6$, then $t = 6(2n + 1)$. So $m = 2n + 1 \in \mathbb{Z}$ exists, and $t \in S$. On the other hand, $12 \in S$ is not of the form $12n + 6$ for any $n \in \mathbb{Z}$.

Some Useful Sets

$$\mathbb{N} = \{0, 1, 2, 3, \dots\}$$

$$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \dots\}$$

Every natural number is an integer:

$$\mathbb{N} \subseteq \mathbb{Z}$$

But there exist integers that are not natural numbers, like -1:

$$\mathbb{N} \subset \mathbb{Z}$$

Examples of sets

- $\{0, 1, 314\}$
- $\emptyset = \{\}$ (the empty set)
- $\{0\}$
- $\{\{\}, \{0, 1, 2, 3\}, \{0\}\}$
- $\{x \in \mathbb{Z} \mid x \text{ is a multiple of } 314\}$
- $\{x \in \mathbb{Z} \mid \exists p,\ (p \text{ is prime} \wedge (x = p + p^2))$
- $\{(x, y) \in \mathbb{R} \mid x^2 + y^2 = 1\}$

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The size of a set is the number of elements in the set.

Examples of sets

	size
$\{0, 1, 314\}$	3
$\emptyset = \{\}$ (the empty set)	0
$\{0\}$	1
$\{\{\}, \{0, 1, 2, 3\}, \{0\}\}$	3
$\{x \in \mathbb{Z} \mid x \text{ is a multiple of } 314\}$	∞
$\{x \in \mathbb{Z} \mid \exists p, \ (p \text{ is prime} \wedge (x = p + p^2))\}$	∞
$\{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x^2 + y^2 = 1\}$	∞

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The size of a set is the number of elements in the set. Sets with a finite number of elements are called finite. Sets with an infinite number of elements are called infinite.

Examples of sets

	size
$\{0, 1, 314\}$	FINITE 3
$\emptyset = \{\}$ (the empty set)	FINITE 0
$\{0\}$	FINITE 1
$\{\{\}, \{0, 1, 2, 3\}, \{0\}\}$	FINITE 3
$\{x \in \mathbb{Z} \mid x \text{ is a multiple of } 314\}$	INFINITE ∞
$\{x \in \mathbb{Z} \mid \exists p, \ (p \text{ is prime} \wedge (x = p + p^2))\}$	INFINITE ∞
$\{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x^2 + y^2 = 1\}$	INFINITE ∞

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A relation is the truth set of a predicate that takes two inputs.

Relations and Maps

A relation is the truth set of a predicate that takes two inputs.

	1	2	3	4	5	6
1	(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)
2	(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	(6,2)
3	(1,3)	(2,3)	(3,3)	(4,3)	(5,3)	(6,3)
4	(1,4)	(2,4)	(3,4)	(4,4)	(5,4)	(6,4)
5	(1,5)	(2,5)	(3,5)	(4,5)	(5,5)	(6,5)
6	(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)

$$S = T = \{1, 2, 3, 4, 5, 6\}$$

$$(x, y) \in S \times T$$

$$P(x, y): \; x + y = 7$$

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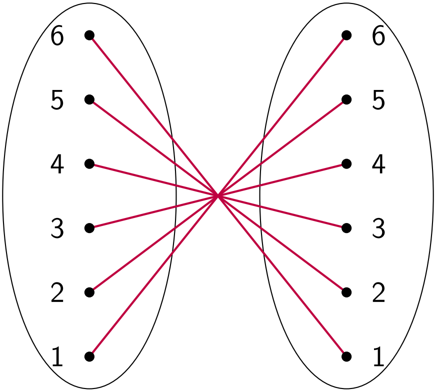

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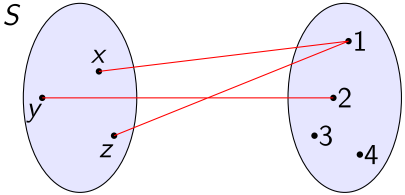
Relations and Maps

A relation is the truth set of a predicate that takes two inputs. A relation could be a function or a map from A to B as long as every element in set A "goes" to exactly one element in the set B.



$A = \{x, y, z\}$ $\qquad$ $B = \{1, 2, 3, 4\}$

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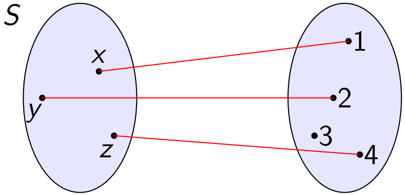

$$A = \{x, y, z\} \qquad B = \{1, 2, 3, 4\}$$

A function/map from A to B that only assigns at most one element of A to each element of B is called one-to-one (1-1).

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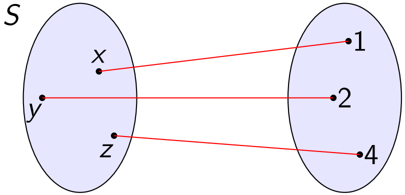

$$A = \{x, y, z\} \qquad B = \{1, 2, 3, 4\}$$

A function/map from A to B assigns at least one element of A to each element of B is called onto.

Relations and Maps

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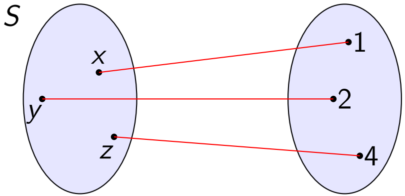
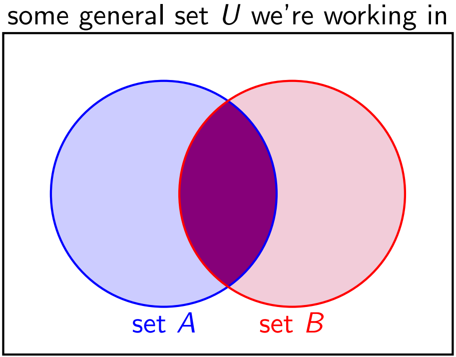


$$A = \{x, y, z\} \qquad B = \{1, 2, 3, 4\}$$

A function/map from A to B that is both one-to-one (1-1) and onto is called a bijection.

Venn Diagrams



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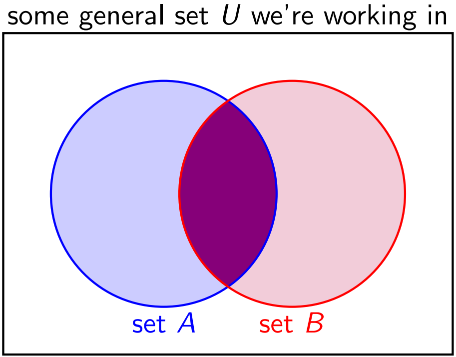

The intersection of A and B

$$A \cap B$$

is the set of elements that belong to both sets.

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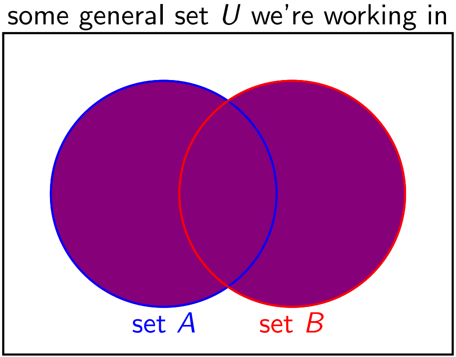

The intersection of A and B

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The union of A and B

$$A \cup B$$

is the set of elements that belong to at least one of the two sets.

Operations on Sets

Intersection:

$$A \cap B$$

The set of elements $x \in U$ such that $x \in A$ and $x \in B$.

Union:

$$A \cup B$$

The set of elements $x \in U$ such that $x \in A$ or $x \in B$.

Complement:

$$A^C$$

The set of elements $x \in U$ such that $x \notin A$.

Set difference:

$$A - B$$

The set of elements $x \in U$ such that $x \in A$ and $x \notin B$.

Operations on Sets

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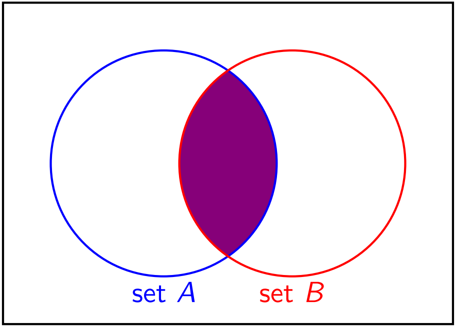

some general set U we're working in

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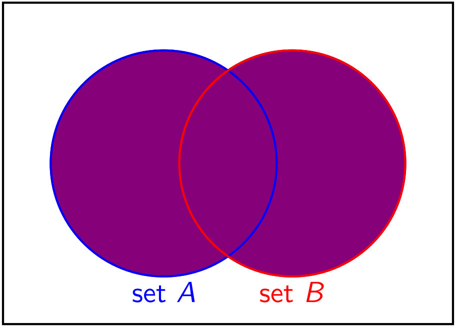

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Operations on Sets

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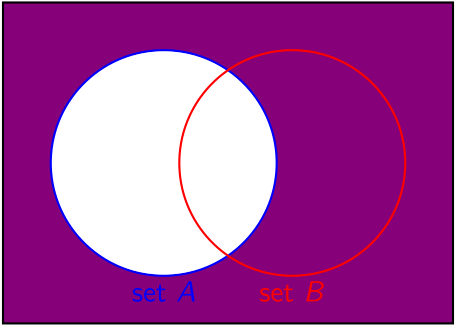

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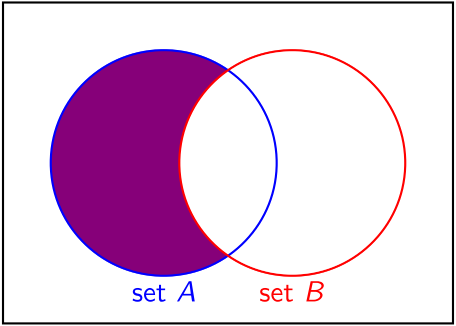

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Observations

For all sets A, B over a universal set U we have:

$$A \cup B = B \cup A \text{ and } A \cap B = B \cap A$$

$$A \cup A = A \cap A = A$$

$$A \cap U = A \cup \emptyset = A$$

$$A \cap B \subseteq B$$

$$(A \subseteq B) \wedge (B \subseteq C) \rightarrow (A \subseteq C)$$

Observations

For a universal set U, and sets A and B in U:

a) $A \cup U = U$
b) $A \cap \emptyset = \emptyset$
c) $(A^C)^C = A$
d) $(A \cup B)^C = A^C \cap B^C \leftarrow$ DeMorgan's Laws!
e) $(A \cap B)^C = A^C \cup B^C \leftarrow$ DeMorgan's Laws!
f) $U^C = \emptyset$
g) $\emptyset^C = U$
h) $A \cup A^C = U$
i) $A \cap A^C = \emptyset$

Notice that these work very similarly to relations in propositional logic. Can you explain why?

Proving that two sets are equal, or one is a subset of the other.

To prove that two sets A and B are equal we need both $A \subseteq B$ and $B \subseteq A$. So we need to show that

- Every element of A is also in B.
- Every element of B is also in A.

There are two types of such problems you might see.

1. A set identity, for example $A \cup (A \cap B) = A$.
2. You may be given definitions of two sets, for example:
   $A = \{m \in \mathbb{N} \mid \exists t \in \mathbb{N},\ m = 6t\}$
   $B = \{n \in \mathbb{N} \mid \exists r, s \in \mathbb{N},\ n = 2r = 3s\}$

We will now see proofs in these two cases.

Proving a set identity

Let A, B be subsets of some unversal set U. Show that

$$A \cup (A \cap B) = A$$

Proof: First we need to show that if $x \in A \cup (A \cap B)$, then $x \in A$. Set union acts like an "or" statement, so we know that at least one of $x \in A$ or $x \in (A \cap B)$ has to be true. If it's the former, we're done. If the latter, we have $(A \cap B) \subseteq A$ and we're also done.

Now we need to show that if $x \in A$, then $x \in A \cup (A \cap B)$. As in an "or" statement, it's enough if one of $x \in A$ or $x \in (A \cap B)$ is true, so we're done! $\square$

Proving a set identity (using contradiction)

Let A, B be subsets of some unversal set U. Show that

$$A \cup (A \cap B) = A$$

Proof: First we need to show that if $x \in A \cup (A \cap B)$, then $x \in A$. Suppose for contradiction that $x \notin A$. $x \in A \cup (A \cap B)$ means "either $x \in A$ or $x \in (A \cap B)$. We assumed that $x \notin A$, so we're left with the $x \in (A \cap B)$ option. But since $A \cap B \subseteq A$ by definition, that is also imposssible! We arrive at a contradiction and conclude that $A \cup (A \cap B) \subseteq A$.

Now we need to show that if $x \in A$, then $x \in A \cup (A \cap B)$. As in an "or" statement, it's enough if one of $x \in A$ or $x \in (A \cap B)$ is true, so we're done! □

Proving inclusion (from set definitions)

Let:
$A = \{m \in \mathbb{N} \mid \exists t \in \mathbb{N},\ m = 6t\}$
$B = \{n \in \mathbb{N} \mid \exists r, s \in \mathbb{N},\ n = 2r = 3s\}$
We want to show that $A \subseteq B$.

Proof: We want to show that for any $m \in A$, we have $m \in B$. So take any $m \in A$. There exists $t \in \mathbb{N}$ such that $m = 6t$. Then $m = 2(3t)$, so $3t$ can serve as r and $m = 3(2t)$, so $2t$ can serve as s. Therefore $m \in B$ for any $m \in B$, and so $A \subseteq B$. $\square$

How would you show that $B \subseteq A$?

Proving inclusion (from set definitions)

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How would you show that $B \subseteq A$? We'll get to that when we do number theory. (Or induction.)

Statements About Sets

Let A, B, C be any sets in a universal set U.

$$A \subseteq B \wedge B \subseteq C^C \rightarrow A \cap C = \emptyset$$

What does this say?

Statements About Sets

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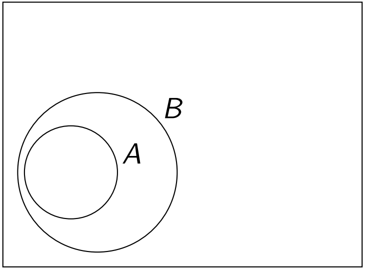
$A \subseteq B$



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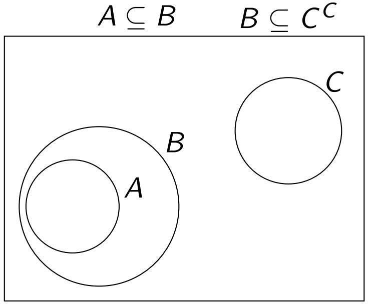

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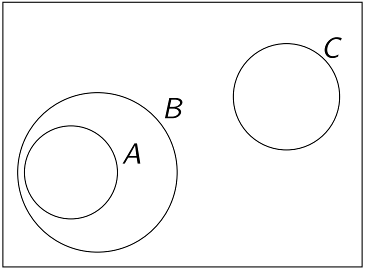

Let A, B, C be any sets in a universal set U.

$$A \subseteq B \wedge B \subseteq C^C \to A \cap C = \emptyset$$

Proof: Assumptions:

$$A \subseteq B, \quad B \subseteq C^C$$

Want:

$$\neg \exists x \in U,\ (x \in A) \wedge (x \in C)$$

Suppose that $x \in A$. Then since $A \subseteq B$ is true, $x \in B$. But then by $B \subseteq C^C$, we have $x \in C^C$ and therefore, $x \notin C$. We get that if $x \in A$, then $\neg(x \in C)$ and therefore $(x \in A) \wedge (x \in C)$ is false. $\square$

Disjoint Sets

Two sets A and B are disjoint if $A \cap B = \emptyset$.

Which of these are pairs of disjoint sets?

- Odd and even integers.
- Odd integers and the empty set.
- Even integers and prime numbers.
- Positive numbers and negative numbers.
- Odd numbers and perfect squares.
- Socks and trees.

Power Set

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- $\{0\}$ 10
- $\{1\}$ 01
- $\{0, 1\}$ 11

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Set Partition

Example: $U = \{1, 2, \ldots, 10\}$

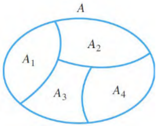
$$A = \{2,4,6,8,10\}, \;\; B = \{1\}, \;\; C = \{3,5,7,9\}$$

Definition:

Sets $A_1, A_2, \ldots$ (possibly infinite) are a **partition** of a set A, if

(a) they are mutually disjoint

(b) their union is equal U.



Example: Universal set $\mathbb{Z}$. Sets

$$A_0 = \{x \in \mathbb{Z}:\;\; \exists k \in \mathbb{Z}, x = 3k\}$$
$$A_1 = \{x \in \mathbb{Z}:\;\; \exists k \in \mathbb{Z}, x = 3k + 1\}$$
$$A_2 = \{x \in \mathbb{Z}:\;\; \exists k \in \mathbb{Z}, x = 3k + 2\}$$

form a partition of $\mathbb{Z}$.

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If you have a drawer full of unpaired socks, and they come in 4 different colors, how many at least do you need to pull out to be sure you have a matching pair?

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1. Show that if seven distinct numbers are selected from $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$, then some two of these numbers sum up to 12.

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2. Show that any collection of eight distinct integers contains distinct integers x and y such that $x - y$ is a multiple of 7.

   What if the "distinct" condition wasn't there?