A few words on Assignment 1

The difference between a proof and a proof by contradiction. We know that $(P \to Q) \land Q$ is true and want to show by contradiction that Q is true.

P	Q	$P \to Q$	$(P \to Q) \land P$	$\neg Q$
1	1	1	1	0
1	0	0	0	1
0	1	1	0	0
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A few words on Assignment 1

- In a normal proof, we cross out the lines where assumptions are false and show that we're only left with the lines where conclusion is true.
- In a proof by contradiction we cross out the lines where conclusion is true and show that we're only left with lines where at least one assumption is false.

MTH314: Discrete Mathematics for Engineers

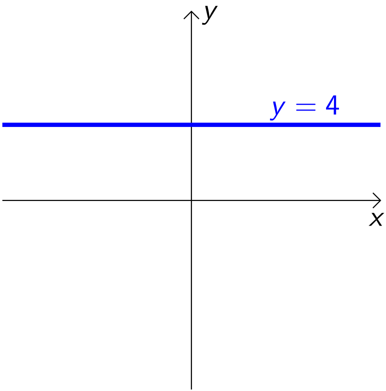
Lecture 2: Predicate Logic

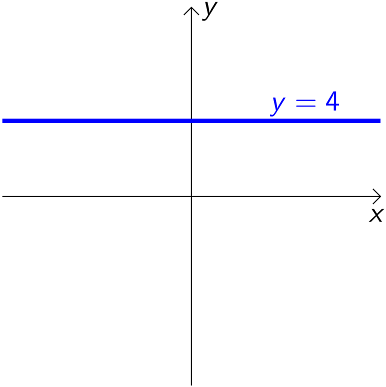
Dr Ewa Infeld

Ryerson Univesity

11 January 2017

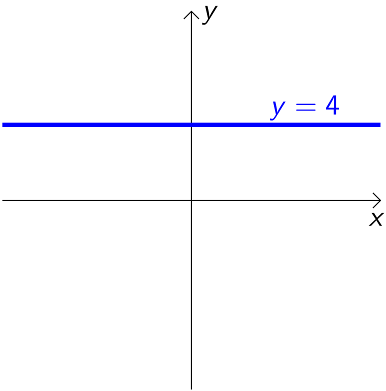
Sets





This plane is a collection of points with coordinates (x, y).

Sets



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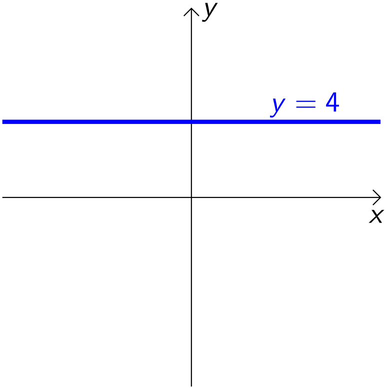

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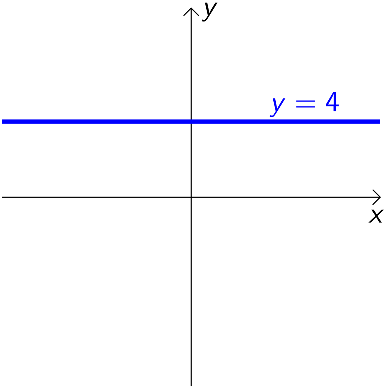
$y = 4$ is a CONDITION that the point needs to fulfill to belong to the line.



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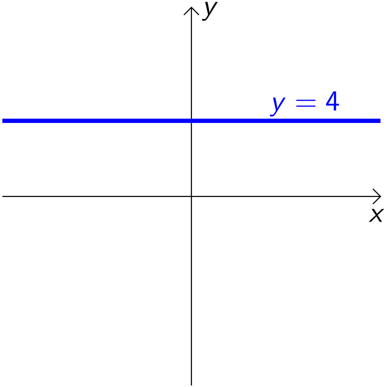

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PREDICATE

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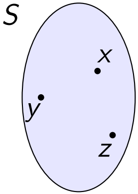
Sets

A set is a collection of objects. It is determined by the elements that belong to it.



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$x \in S$ reads as "x is an element of S." or "x belongs to S."

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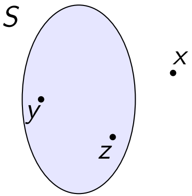

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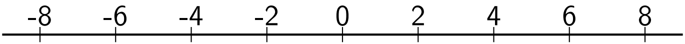
A set is a collection of objects. It is determined by the elements that belong to it.

We think of two sets that have the same elements as the same set.

Example:
The set of natural numbers that are multiples of 2, and the set of even numbers are the same set.



Why does it matter?

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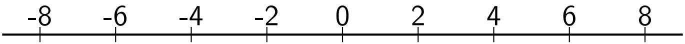


Why does it matter? If two different programs compute the same thing, are they the same program?

Set Notation

$x \in S$ reads as "x is an element of S." or "x belongs to S."

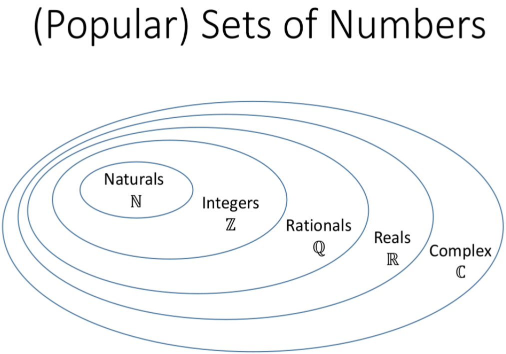
$x \notin S$ reads as "x is not an element of S." or "x does not belong to S."

If the set S is a set of breakfast options, and you can pick eggs, oatmeal or fruit, we use this notation:

$$S = \{\text{eggs, oatmeal, fuit}\}$$

Sometimes we see ":=" as in, $S := \{eggs, oatmeal, fuit\}$. This usually happens when you *define* something. You can think of a parallel with programming - the first time you declare S to be something ($S := \{eggs, oatmeal, fuit\}$), vs when you simply state a fact about S, ($S = \{eggs, oatmeal, fuit\}$.) You don't always need to "declare" it in math though.

Some Useful Sets



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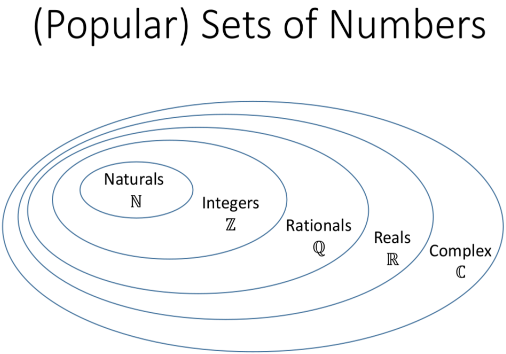

$$\mathbb{N} = \{0, 1, 2, 3, \dots\}$$

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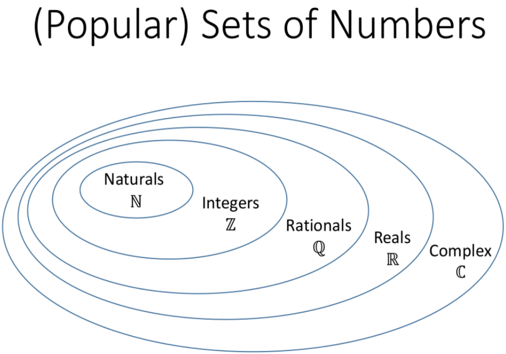


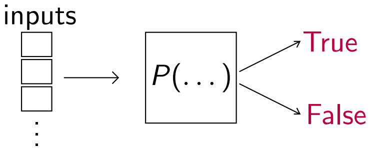
$\mathbb{N} = \{0, 1, 2, 3, \dots\}$ $\quad$ $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \dots\}$

Predicates

A *predicate* is a sentence with a finite number of variables that becomes a *statement* when specific values (from a set) are substituted for the variables. Then it is either *true* or *false.*

Example: Take set $S = \{\text{eggs, oatmeal, fuit}\}$ Predicate $P(x)$: Alice had x for breakfast.

If we know that Alice had oatmeal and fruit for breakfast, $P(x)$ evaluates as true on $x = \textit{oatmeal}$ or $x = \textit{fruit}$ and false on $x = \textit{eggs}$.




```
def P(x,y):
    if x>y:
        return True
    else:
        return False
```

Truth Set of a Predicate

A predicate $P(x)$ evaluated on set S has a *truth set*, that is, all the values $x \in S$ on which P evaluates as *true*. The truth set of $P(x)$ is a *subset* of S. Write $\{x \in S | P(x)\}$ for the truth set. It reads as "The set of x in S such that $P(x)$."

If the predicate $P(x)$ is false for every $x \in S$, the truth set is the *empty set*.

$$\{x \in S | P(x)\} = \{\} = \emptyset$$

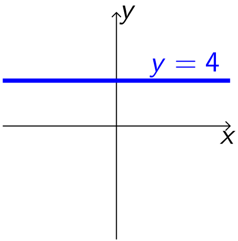
If the predicate $P(x)$ is true for every $x \in S$, the truth set is S itself.

$$\{x \in S | P(x)\} = S$$

Another set T is a *subset* of S if every element of T is also an element of S. The emty set and S itself are both subsets of S.

Some Predicates and their Truth Sets

We're working in the Cartesian plane,
$\mathbb{R} \times \mathbb{R} = \{(x, y) \mid x \in \mathbb{R},\ y \in \mathbb{R}\}$



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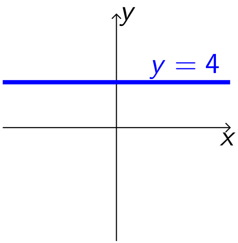

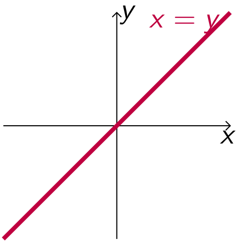

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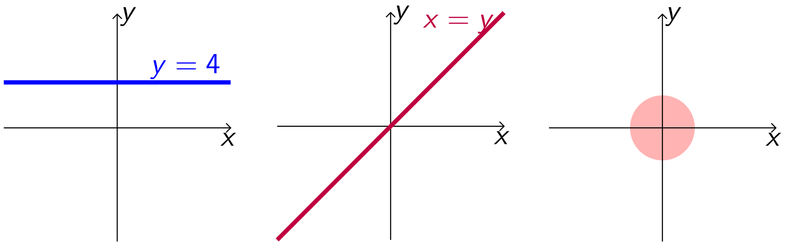

$P(x, y): \ y = 4$ $\qquad$ $Q(x, y): \ x = y$ $\qquad$ $R(x, y): \ x^2 + y^2 < 1$

Cartesian Product of Sets

If S and T are sets, and we need to pick an element from each, we are really thinking of a set $S \times T$, a *Cartesian product* of S and T.

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5	(1,5)	(2,5)	(3,5)	(4,5)	(5,5)	(6,5)
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$$S = T = \{1, 2, 3, 4, 5, 6\}$$

$$(x, y) \in S \times T$$

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Quantifiers

$\forall x \in S$ reads as "for all x in set S..."
$\exists x \in S$ reads as "there exists x in set S..."

Example:
Let H be the set of human beings.
$P(x)$: x is mortal.

$$\forall x \in H,\ P(x)$$

means "All human beings are mortal."

$$\exists x \in H, P(x)$$

means "There exists a human being that is mortal."

What are negations of these statements?

Quantifiers

Example:
Let H be the set of human beings.
$P(x)$: x is mortal.

$\forall x \in H,\ P(x)$ | $\exists x \in H,\ \neg P(x)$

"All human beings are mortal." | "There exists a human being that's immortal."

$\exists x \in H, P(x)$ | $\forall x \in H,\ \neg P(x)$

"There exists a human being that is mortal." | All human beings are immortal.

What are negations of these statements?

Quantifiers

Three logicians walk into a bar. The bartender asks “Does everyone want beer?”
The first logician says “I don’t know.”
The second logician says “I don’t know.”
The third logician says “Yes.”

What happened here?

What if the bartender asked “Does anyone want beer?”

Statements with Multiple Quantifiers

"There is no smallest positive real number"

$\forall$ positive real numbers x, $\exists$ a positive real number y such that $y < x$.

We sometimes say "strictly positive" to emphasise that we don't include 0.

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$P(y): \ y > 0$
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Statements with Multiple Quantifiers

"There is no smallest positive real number"

$\forall$ positive real numbers x, $\exists$ a positive real number y such that

$$y < x.$$

$P(x): \ x > 0$, let $\mathbb{R}^+ := \{x \in \mathbb{R} | P(x)\}$
$Q(x, y): \ y < x$

$$\forall x \in \mathbb{R}^+, \ \exists y \in \mathbb{R}^+ : \ Q(x, y)$$

We sometimes say "strictly positive" to emphasise that we don't include 0.

Negations of Statements with Multiple Quantifiers

We know that:

$$\neg(\forall x \in S,\ P(x)) \equiv \exists x \in S,\ \neg P(x)$$

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So what is

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"It is not true that for all x in S there exists y in T such that $P(x,y)$ is true."

" There exists x in S

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GO SYSTEMATICALLY FROM LEFT TO RIGHT

Negations of Statements with Multiple Quantifiers

Example:
Let H be the set of human beings.
Let $F(x, y)$ be the predicate that means that $x, y \in H$ are friends.

$$\forall x \in H,\ \exists y \in H, P(x, y)$$

"Every person x has a friend y."

$$\exists x \in H,\ \forall y \in H,\ P(x, y)$$

"There exists a person x that is friends with everyone."

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Vacuous Truth

Suppose we have an expression $P \to Q$, and we know P must be false. Then $P \to Q$ is *vacuously* true.

Vacuous means "empty." We haven't actually learned anything about Q.

For statements with quantifiers, this means every statement about the empty set is true.

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For statements with quantifiers, this means every statement about the empty set is true.

U is the set of unicorns.
$P(x)$ is "x is pink."

$$\forall x \in U,\ P(x)$$

"All unicorns are pink."

Vacuous Truth

Suppose we have an expression $P \to Q$, and we know P must be false. Then $P \to Q$ is *vacuously* true.

Vacuous means "empty." We haven't actually learned anything about Q.

For statements with quantifiers, this means every statement about the empty set is true.

U is the set of unicorns.
$P(x)$ is "x is pink."

$$\forall x \in U,\ P(x)$$

"All unicorns are pink." TRUE

Vacuous Truth

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A STATEMENT ABOUT THE EMPTY SET IS VACUOUSLY TRUE. THINK OF THE IMPLICATION WHERE THE CONDITION IS FALSE AS A STATEMENT ABOUT THE EMPTY SET OF CASES WHERE IT IS TRUE.

Quantified Conditional Statements

Let S be the set of Ryerson students.
$P(x):$ x is a MTH314 student.
$Q(x):$ x brought an iclicker to class today.

$$\forall x \in S,\ P(x) \to Q(x)$$

"All MTH314 students brought iclickers to class today."

Quantified Conditional Statements

Let S be the set of Ryerson students.
$P(x)$: x is a MTH314 student.
$Q(x)$: x brought an iclicker to class today.

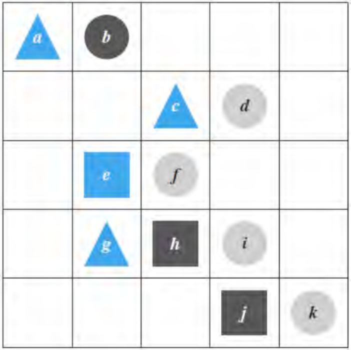
$$\forall x \in S,\ P(x) \rightarrow Q(x)$$

"All MTH314 students brought iclickers to class today."

If this is true, $P(x)$ is a sufficient condition for $Q(x)$ on domain S.

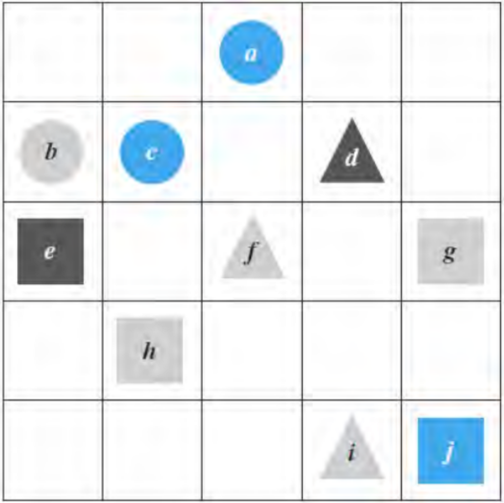
And $Q(x)$ is a necessary conditions for $P(x)$.

Tarski's World: Domain $U = \{a, b, \ldots, k\}$



Which of the claims below are true (and why)?

- $\forall t \in U, Triangle(t) \rightarrow Blue(t)$
- $\forall t \in U, Blue(t) \rightarrow Triangle(t)$
- $\exists t \in U, Square(t) \rightarrow RightOf(d, t)$
- $\exists t \in U, Square(t) \wedge Gray(t)$


a
b
c
d
e
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h
i
j

Proof Arguments

Suppose that:

$$(P \vee Q) \vee R$$

$$\neg P$$

$$Q \rightarrow R$$

We would like to prove R, not by writing out a truth table, but by a mathematical argument.

Proof by contradiction:

Proof Arguments

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We would like to prove R, not by writing out a truth table, but by a mathematical argument.

Proof by contradiction: Suppose $\neg R$. We know that $Q \rightarrow R$, so we must have $\neg Q$.

Proof Arguments

Suppose that:

$$(P \vee Q) \vee R$$

$$\neg P$$

$$Q \to R$$

We would like to prove R, not by writing out a truth table, but by a mathematical argument.

Proof by contradiction: Suppose $\neg R$. We know that $Q \to R$, so we must have $\neg Q$. But then we have all $\neg P$, $\neg Q$, and $\neg R$ so $(P \vee Q) \vee R$ must be false. But it was one of the assumptions, so this cannot work!

Proof Arguments

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$$\neg P$$

$$Q \rightarrow R$$

We would like to prove R, not by writing out a truth table, but by a mathematical argument.

Proof by contradiction: Suppose $\neg R$. We know that $Q \rightarrow R$, so we must have $\neg Q$. But then we have all $\neg P$, $\neg Q$, and $\neg R$ so $(P \vee Q) \vee R$ must be false. But it was one of the assumptions, so this cannot work!

Therefore, as evidenced by a proof by contradiction, R is true. □

Proofs with Quantifiers

Proof that the sum of every two even integers is even.

Definition: an integer n is even if and only if there exists an integer r such that $n = 2r$.

We're working in set (domain) $\mathbb{N}$.
$P(n, r):\ n = 2r$ is a predicate (and a relation!)

Suppose $m,\ n$ are even.
Want: $m + n$ is even.

$\exists r \in \mathbb{N}, P(n, r)$ $\qquad n = 2r$
$\exists s \in \mathbb{N}, P(m, s)$ $\qquad m = 2s$

$$n + m = 2r + 2s$$
$$n + m = 2(r + s)$$

Since $r + s$ is an integer, $n + m$ must be even. $\square$

Proofs with Quantifiers

"There is no smallest positive real number"
$\forall$ positive real numbers x, $\exists$ a positive real number y such that $y < x$.

$$P(x): \; x > 0, \text{ let } \mathbb{R}^+ := \{x \in \mathbb{R} | P(x)\}$$
$$Q(x,y): \; y < x$$

$$\forall x \in \mathbb{R}^+, \; \exists y \in \mathbb{R}^+: \; Q(x,y)$$

Proofs with Quantifiers

"There is no smallest positive real number"
$\forall$ strictly positive real numbers x, $\exists$ a positive real number y such that $y < x$.

$$P(x): \; x > 0, \text{ let } \mathbb{R}^+ := \{x \in \mathbb{R} | P(x)\}$$
$$Q(x, y): \; y < x$$

$$\forall x \in \mathbb{R}^+, \; \exists y \in \mathbb{R}^+ : \; Q(x, y)$$

Proof: Let x be any strictly positive real number. Then $y = \frac{x}{2}$ is a strictly positive real number that is smaller than x. Therefore:

$$\forall x \in \mathbb{R}^+, \; \exists y \in \mathbb{R}^+ : \; Q(x, y).$$

□