MTH314: Discrete Mathematics for Engineers

Lecture 1: Propositional Logic

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Logistics

Section 2:

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Evaluation:

iClickers	15%+5% bonus	starting week 2
Midterm I	20%	on February 15
Midterm II	25%	on March 22
Final Exam	40%	final exam period

Logistics

Online resources:

- d2l
- Discussion board on piazza.com

Your best resources to succeed in the class:

1.
2.
3.

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Let's get started!

Today, we will try to understand logic gates that every processor or memory module is made of.

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We have a statement P and assign it a value 1 if P is true and 0 if P is false.

Example: P : "I am 21 years old."

Truth table (depicting all the possibilities):

P
0
1

A more elaborate truth table.

P : "I am 21 years old."

Q : "I play soccer."

P	Q
0	0
0	1
1	0
1	1

Which row of the table corresponds to you?

Negation

Statement: a declarative sentence that is either true or false. We will denote statements with capital letter, for example $P, \ Q, \ R \dots$

The *negation* $\neg P$ of a statement P is a statement that evaluates to true if P evaluates to false, and evaluates to false if P evaluates to true. (Some textbooks write $\sim P$ or "NOT P" instead.)

Example:

$$P : \text{"Earth is flat."}$$
$$\neg P : \text{"Earth is not flat."}$$

P	$\neg P$
0	1
1	0

Conjunction and Disjunction

Conjunction (AND) is true if and only if both statements are true.

$$P \wedge Q$$

Disjunction (OR) of two statements is true if at least one of the statements is true.

$$P \vee Q$$

P : "I am 21 years old."
Q : "I play soccer."

$P \wedge Q$: "I am 21 years old AND I play soccer."
$P \vee Q$: "I am 21 years old OR I play soccer."

Remember: the Disjunction symbol points Down. And the symbol for Conjunction looks a bit like the "A" in AND.

Conjunction and Disjunction

P : "I am 21 years old."
Q : "I play soccer."

$P \wedge Q$: "I am 21 years old AND I play soccer."
$P \vee Q$: "I am 21 years old OR I play soccer."

Draw a truth table for these.

P	Q	$P \wedge Q$	$P \vee Q$
0	0		
0	1		
1	0		
1	1		

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1	0	0	1
1	1	1	1

Implication and Equivalence

P : "I am 21 years old."
Q : "I play soccer."

$P \rightarrow Q$: "IF I am 21 years old THEN I play soccer."
(IMPLICATION)

$P \leftrightarrow Q$: "I am 21 years old IF AND ONLY IF I play soccer."
(DOUBLE IMPLICATION/EQUIVALENCE)

Implication and Equivalence

P : "I am 21 years old."
Q : "I play soccer."

$P \to Q$: "IF I am 21 years old THEN I play soccer."
$P \leftrightarrow Q$: "I am 21 years old IF AND ONLY IF I play soccer."

P	Q	$P \to Q$	$P \leftrightarrow Q$
1	1	1	1
1	0	0	0
0	1	1	0
0	0	1	1

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Unary and binary operators.

```
def BinaryAddition(x,y):
    return x+y
    print "I require two inputs."

def UnaryAddition(x):
    return x+7
    print "I require one input."
```

The negation operator is unary.
The conjunction, disjunction, implication and equivalence operator are binary, they need exactly two inputs.

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$$P \vee Q \vee R$$

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$$(P \vee Q) \vee R$$

Combining logical operators.

We evaluate $\neg$ before the binary operators, that is:

$$\neg P \vee Q \equiv (\neg P) \vee Q,$$

as opposed to:

$$\neg(P \vee Q).$$

P	$\neg P$	Q	$\neg P \vee Q$	$\neg P \to Q$	$(\neg P \vee Q) \leftrightarrow (\neg P \to Q)$
1	0	1			
1	0	0			
0	1	1			
0	1	0			

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1	0	1	1		
1	0	0	0		
0	1	1	1		
0	1	0	1		

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1	0	1	1	1	
1	0	0	0	1	
0	1	1	1	1	
0	1	0	1	0	

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P	$\neg P$	Q	$\neg P \vee Q$	$\neg P \to Q$	$(\neg P \vee Q) \leftrightarrow (\neg P \to Q)$
1	0	1	1	1	1
1	0	0	0	1	0
0	1	1	1	1	1
0	1	0	1	0	0

Remarks.

Just as $\neg$ comes before $\wedge$ and $\vee$,
$\wedge$ and $\vee$ come before $\rightarrow$ and $\leftrightarrow$.

Two logical statements for which their truth tables agree are called logically equivalent.
A logical statement that evaluates always to true is called a tautology.
A logical statement that evaluates always to false is called a contradiction.

P is a tautology if and only if $\neg P$ is a contradiction.
P and Q are logically equivalent if and only if $P \leftrightarrow Q$ is a tautology.

Examples of tautologies

- $P \vee \neg P$
- $(P \to Q) \vee P$
- De Morgan's Laws:
  - $\neg(P \vee Q) \leftrightarrow \neg P \wedge \neg Q$
  - $\neg(P \wedge Q) \leftrightarrow \neg P \vee \neg Q$

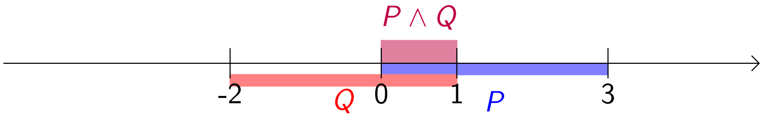
Examples of tautologies

- $P \vee \neg P$
- $(P \rightarrow Q) \vee P$
- De Morgan's Laws:
  - $\neg(P \vee Q) \leftrightarrow \neg P \wedge \neg Q$
  - $\neg(P \wedge Q) \leftrightarrow \neg P \vee \neg Q$

Example:

P : x is a real number greater than 0 but smaller than 3.

Q : x is a real number smaller than 1 but greater than -2.



Examples of contradictions

- $P \wedge \neg P$
- $(P \to Q) \wedge (P \wedge \neg Q)$
- $P \vee Q \leftrightarrow \neg P \wedge \neg Q$
- $P \wedge Q \leftrightarrow \neg P \vee \neg Q$

Some important math expressions

For some logical statements $P,\ Q$:

"P is a sufficient condition for Q."
"P is true only if Q."
means $P \to Q$ is a tautology.

"P is a necessary condition for Q."
"P is true if Q."
means $Q \to P$ is a tautology.

"P is a necessary and sufficient condition for Q"
"P is true if and only if Q"
"P is equivalent to Q."
means $P \leftrightarrow Q$ is a tautology.

Some important math expressions

For some logical statements $P,\ Q$:

"P is a sufficient condition for Q."
"P is true only if Q."
means $P \to Q$ is a tautology.

Example:
P : "This is a Tesla Model X"
Q : "This is a car."
We have $P \to Q$, because any Tesla Model X is a car. However, not every car is a Tesla Model X, so while P is a sufficient condition for Q, it is not necessary.
"This is a Tesla Model X only if it is a car."

Can you come up with a sentence in English (rather than math) for each of these tautologies?

"$\equiv$" reads as "is equivalent to", "is the same as" or simply "is."

- $\neg(\neg P) \equiv P$ (double negative)
- $P \leftrightarrow Q \equiv (P \to Q) \land (Q \to P)$
- $P \lor$ false $\equiv P$
- $P \land \neg P \equiv$ false
- $P \lor (Q \land R) \equiv (P \lor Q) \land (P \lor R)$ (distributive law)
- $P \lor Q \equiv Q \lor P$ (commutative law)
- $P \to (Q \to R) \equiv Q \to (P \to R)$

What can we do with an implication?

Example:
P : It is Monday.
Q : There is no MTH314 class.

We know that $P \to Q$ is true. What about the following statements?

Converse: $Q \to P$

Inverse:: $\neg P \to \neg Q$

Contrapositive: $\neg Q \to \neg P$

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Contrapositive: $\neg Q \to \neg P$ TRUE

What is a proof?

Start with some statements you assume to be true and use the properties of logical operations to show that something else is true.

It's a bit like arguing a case in court. You start with some assumptions, like laws and evidence, and make a step-by-step, air-tight argument to convince the judge of your conclusion.

Example: Suppose you know that both $P \to Q$ and P are true. Then Q is true. You can use a truth table to convince a picky judge.

What is a proof?

Example: Suppose you know that both $P \to Q$ and P are true. Then Q is true.

P	Q	$P \to Q$
1	1	1
1	0	0
0	1	1
0	0	1

What is a proof?

Example: Suppose you know that both $P \to Q$ and P are true. Then Q is true.

$P \to Q$ is true.

P	Q	$P \to Q$
1	1	1
~~1~~	~~0~~	~~0~~
0	1	1
0	0	1

What is a proof?

Example: Suppose you know that both $P \rightarrow Q$ and P are true. Then Q is true.

$P \rightarrow Q$ is true.
P is true.

P	Q	$P \rightarrow Q$
1	1	1
~~1~~	~~0~~	~~0~~
~~0~~	~~1~~	~~1~~
~~0~~	~~0~~	~~1~~

What is a proof?

Example: Suppose you know that both $P \to Q$ and P are true. Then Q is true.

$P \to Q$ is true.
P is true.

P	Q	$P \to Q$
1	1	1
~~1~~	~~0~~	~~1~~
~~0~~	~~1~~	~~0~~
~~0~~	~~0~~	~~1~~

What is a proof?

An argument of this form:

$P \to Q$, P
$\quad \therefore Q$

Is called "Modus Ponens." (The $\therefore$ symbol reads as "therefore.")

We call $P \to Q$ and P the *premises* and Q the *conclusion*.

You can similarly verify other arguments:
$P \to Q$, $\neg Q$
$\quad \therefore \neg P$

What about
$P \to Q$
$\quad \therefore Q \to P$?

Proof by Contradiction

Proof by contradiction is of this form:

$\neg P \rightarrow$ false

$\quad \therefore P$

Example: P : Every Tesla Model X is a car.

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If we can argue that $\neg P$ is false, we conclude that P is true.

P	$\neg P$	false
1	0	0
0	1	0

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P	$\neg P$	false	
1	0	0	
~~0~~	~~1~~	~~0~~	$\neg P \rightarrow$ false