Practice Midterm Exam

Math 20

October 2014

Name: ____

Please read the following instructions before starting the exam:

- This exam is closed book. You may not give or receive any help during the exam, though you may ask the instructors for clarification if necessary.
- Be sure to show all work whenever possible. Even if your final answer is incorrect, an appropriate amount of partial credit can be assigned.
- Please circle or otherwise indicate your final answer, if possible.
- The test has a total of 8 questions, worth a total of 100 points. Point values are indicated for each question.
- You will have two hours from the start of the exam to complete it.
- Good luck!

HONOR STATEMENT: I have neither given nor received help on this exam, and I attest that all the answers are my own work.

SIGNATURE:______________________

DOES NOT APPLY TO PRACTICE EXAMS

This page is for grading purposes only

Problem	Points
1	
2	
3	
4	
5	
6	
7	
8	
Total	

1. **Multiple choice.** Circle the correct answer to each question. [5 points each, no partial credit].

Let random variables X, Y each be picked from [0,1] with uniform density. Which of the following events is independent of $x < y$?

(a) $(x-1)^2 + y^2 \leq 1$

(b) $x^2 + y^2 \leq 1$

(c) $y < 2x$

(d) $y < x$

Which is the correct Stirling's Formula?

(a) $n! \sim (ne)^{-n}\sqrt{2\pi n}$

(b) $n! \sim n^n e^{-n}\sqrt{2\pi n}$

(c) $n! \sim (ne)^{-n}\sqrt{4\pi n}$

(d) $n! \sim n^n e^{-n}\sqrt{4\pi n}$

You are about to play a racing game against your friend, and you agree to play 6 games. He's been practicing a lot more, so in any single game, the probability that you win is 0.45, and the probability that your friend wins is 0.55. What is the chance you win at least 4 times?

(a) $0.45^4 0.55^2 + 0.45^5 0.55 + 0.45^6$

(b) $\binom{6}{4} 0.45^4 0.55^2$

(c) $\binom{6}{4} 0.45^4 0.55^2 + \binom{6}{5} 0.45^5 0.55 + 0.45^6$

(d) 0.45

Pick a point on a circular target with unit radius in the following way: pick a number r from $(0, 1)$ uniformly at random, and a number θ from $(0, 2\pi)$ uniformly at random. The chosen point has polar coordinates (r, θ). What is the probability that this point is within distance 0.25 from the edge of the target?

(a) $1/\sqrt{2}$

(b) $1/2$

(c) $1/4$

(d) $1/2\sqrt{2}$

2. Let the random variable X be the number of rolls of a six-sided die until, and including, the first time 6 comes up.

   - What is the probability distribution of X? [5 points]
   - What is $\mathbb{E}[X]$? [5 points]

3. Find the cumulative probability distributions $F(x) = P[X \leq x]$ for each of the following probability densities: [15 points]

   - $f(x) = \lambda e^{-\lambda x}$, $x \in (0, \infty)$
   - $f(x) = 1/a, \; x \in (0, a)$
   - $f(x) = 1/x^2, \; x \in (1, \infty)$

4. You throw $6n$ six-sided dice. What is the probability that each of $\{1, 2, 3, 4, 5, 6\}$ appears exactly n times? [10 points]

5. Pick a permutation of $\{1, 2, 3, 4, 5, 6\}$ uniformly at random. What is the probability that it has a fixed point that is an even number? [15 points]

6. Recall that the Poisson distribution is:

$$P(X = k, \lambda) = \lim_{n \to \infty} \binom{n}{k} (\lambda/n)^k (1 - \lambda/n)^{n-k}.$$

Use the fact that $P(X = 0, \lambda) = e^{-\lambda}$ to derive an expression for $P(X = k, \lambda)$ that is independent of n. [10 points]

7. Let X and Y be independent random variables with density functions:

   - $f_X(x) = 3/x^4,\ \Omega_X = (1, \infty)$
   - $f_Y(y) = e^{-y},\ \Omega_Y = (0, \infty)$

   Find the expectation and variance of $Z = X + Y$. [10 points]

8. Let X be a continuous random variable that is uniformly distributed on $(0, 2)$. Find the sample space, probability density and cumulative probability distribution of

$$Y = X^3.$$

[10 points]