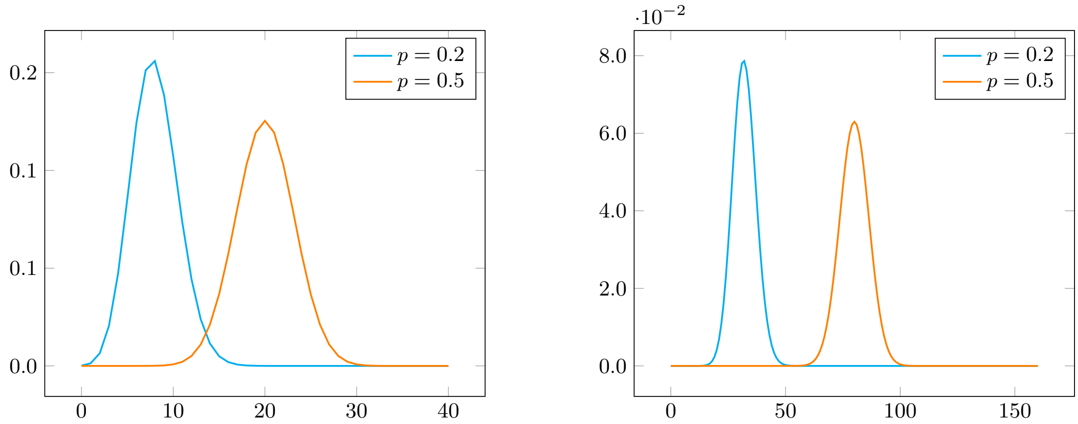
LECTURE 6 NOTES **Math 20 Fall 2014, Dartmouth College**

Contents

1 Binomial Distribution at a glance

As n grows, the diagram will appear smoother, and more strongly centered around $n/2$. In general, even if p is not $1/2$, it will be centered around $p \times n$. The diagram on the left shows the distributions for $n = 40$, with $p = 1/2$ and $p = 0.2$. The diagram on the right shows the same for $n = 160$.



2 Binomial expansion

Binomial coefficients come up in expressions like $(x+1)^n$. This can be expanded as $(x+1)^n = (x+1)(x+1)\dots(x+1)$.

If we start multiplying these out, each term will correspond to a way of picking either x or 1 from each bracket. Its power of x will be the number of times x was chosen. There are $\binom{n}{j}$ ways to choose x j times and 1 $n-j$ times. That's why:

$$(x+1)^n = x^n + nx^{n-1} + \binom{n}{n-2}x^{n-2} + \binom{n}{n-3}x^{n-3} + \dots + \binom{n}{3}x^3 + \binom{n}{2}x^2 + nx + 1.$$

This can be easily testes on $(x+1)^4 = (x+1)(x+1)(x+1)(x+1) = x^4 + 4x^3 + 6x^2 + 4x + 1$.

3 Hypothesis testing

Alice is a good chess player, and her friends have seen her win about 60% of her games against the computer. Recently she's been practicing a lot, and she's pretty sure that her probability of winning a game is now 0.7. She wants to convince her friends of that, so she's planning to play 10 games. Her friends will be convinced if she wins at least 7 of them. Is that a good method?

Null Hypothesis: Alice still wins with probability 0.6.

Alternate Hypothesis: Alice improved her game, and now wins with probability 0.7.

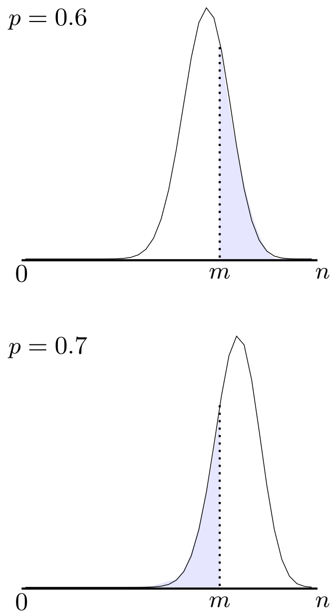


Fig. 1 Probability of type I error for a given m *provided the hypothesis is false* can be pictured on a binomial distribution for $p = 0.6$ (top). This is the probability Alice, *provided she did not improve at chess* still convinces her friends that she did. If she did improve, the probability of type II error for a given m can be pictured on a binomial distribution for 0.7 (bottom). This happens if she didn't win enough games to convince her friends.

If Alice still wins every game with probability 0.6, how would you calculate the probability that she will win at least 7 games? If the alternate hypothesis is false, but a test suggests it's true it's called a *type I error* in statistics.

If Alice now wins every game with probability 0.7, how would you calculate the probability that she wins at most 6 games? If the alternate hypothesis is true, but the experiment suggests that it's false that's called a *type II error*.

Is there some better number m, rather than 7, to set as the threshold? The higher the m, the lower the probability of a type I error, and the higher the probability of a type II error. The key to a good hypothesis test is finding m that makes the probabilities of both errors as small as possible.

Consider a binomial distribution for $p = 0.6$. If Alice didn't get better at chess, then the probability of type I error is:

$$\sum_{i=7}^{10} \binom{10}{i} 0.6^i 0.4^{10-i} \simeq 0.382$$

and if she has improved to $p = 0.7$, the likelihood of type II error is:

$$\sum_{i=0}^{6} \binom{10}{i} 0.7^i 0.3^{10-i} \simeq 0.350$$

The higher the m, the lower the probability of a type I error, and the higher the probability of a type II error. What happens to the errors as we increase the number of games Alice will play? Say, if she plays $n = 20$ games and needs to win 14 of them?

As we vary the number of games n and the success threshold m, the probability of type I error if Alice has not improved is:

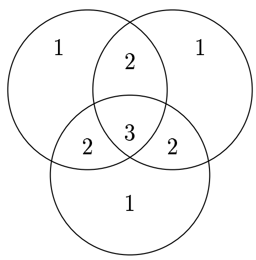
$$\sum_{i=m}^{n} \binom{n}{i} 0.6^i 0.4^{n-i},$$

and the probability of type II error if Alice did improve is:

$$\sum_{i=0}^{m} \binom{n}{i} 0.7^i 0.3^{n-i}.$$

4 Inclusion-exclusion

Recall that for two events A and B, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, since as you add the probabilities of A and B, you add their intersection twice, so you need to subtract it once.



For three events A, B, C we can find an analogous formula. As we add $P(A)$, $P(B)$ and $P(C)$ we include each of the intersections of only 2 sets twice, and the intersection of all 3 three times. If we then subtract each intersection of two sets, we ill subtract the intersection of all three each time, and there are 3 of them. So we need to add $P(A \cap B \cap C)$ in at the end. The overall formula is:

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).$$

The general expression for n events $A_1, A_2, \ldots, A_n$ is:

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = \sum_{1 \le i \le n} P(A_i) - \sum_{1 \le i < j \le n} P(A_i \cap A_j) + \sum_{1 \le i < j < k \le n} P(A_i \cap A_j \cap A_k)$$
$$- \sum_{1 \le i < j < k < l \le n} P(A_i \cap A_j \cap A_k \cap A_l) + \cdots + (-1)^{n+1} P(A_1 \cap A_2 \cap \cdots \cap A_n)$$

So, first adding all single sets, then subtracting intersections of pairs, adding intersections of triples, subtracting intersections of quadruples and so on.

5 Example: fixed points of a permutation

A *fixed point* in a permutation of $\{1, 2, \ldots, n\}$ ia any number j that appears in jth position. For example, if a permutation starts with 1, then 1 is a fixed point. If 2 appears in 2nd position, then 2 is a fixed point. Here are the permutations of $\{1, 2, 3\}$ with their fixed points highlighted in red:

$$123,\ 132,\ 213,\ 231,\ 321,\ 312$$

Pick a permutation of $\{1, 2, \ldots, n\}$ uniformly at random.

- What is the probability that it has n fixed points?
  There is only one permutation that has n fixed points, and that is $12 \ldots n$. The probability is therefore $1/n!$.
- What is the probability that it has $n-1$ fixed points?
  This is impossible - if you fix $n-1$ elements, the last one must also fall into place.
- What is the probability that it has $n-2$ fixed points?
  There are $\binom{n}{2}$ ways to pick two elements to switch. All other elements are fixed. The probability is $\binom{n}{2}/n!$
- What is the probability that 1 is a fixed point of that permutation?
  One way to think about it is that, if you fix 1 you have $(n-1)!$ ways of arranging everything else. So the probability is $(n-1)!/n! = 1/n$. Another way of thinking about it is that there is equally many permutations starting with each number, so necessarily exactly $1/n$ of them start with a 1. The same argument works with any other fixes point, i.e. the probability that i, where $1 \leq i \leq n$, is a fixed point is $1/n$.
- What is the probability that it has no fixed points?

Finding a probability that the permutation has no fixed points is more complicated. Define A_i to be the event that i is a fixed point of the permutation. Then:

$$P(\text{no fixed points}) = 1 - P(\text{at least one fixed point}) = 1 - P(A_1 \cup A_2 \cup \cdots \cup A_n)$$

so we need to find $P(A_1 \cup A_2 \cup \cdots \cup A_n)$. For that, we need inclusion-exclusion.

We know that:

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = \sum_{1 \leq i \leq n} P(A_i) - \sum_{1 \leq i < j \leq n} P(A_i \cap A_j) + \sum_{1 \leq i < j < k \leq n} P(A_i \cap A_j \cap A_k)$$
$$- \sum_{1 \leq i < j < k < l \leq n} P(A_i \cap A_j \cap A_k \cap A_l) + \cdots + (-1)^{n+1} P(A_1 \cap A_2 \cap \cdots \cap A_n).$$

We can find $\sum_{1 \leq i \leq n} P(A_i)$ as follows: we just argued that for any A_i, $P(A_i) = \frac{1}{n}$. There are n of them, so:

$$\sum_{1 \leq i \leq n} P(A_i) = \frac{n}{n} = 1.$$

$A_1 \cap A_2$ is the event that both 1 and 2 are fixed points. If 1 and 2 are fixed, there are still $(n-2)!$ ways to arrange all the other elements. This probability is therefore $\frac{(n-2)!}{n!} = \frac{1}{n(n-1)}$. For any two elements, the probability that they are both fixed is also $\frac{1}{n(n-1)}$. There are $\binom{n}{2}$ such pairs, so:

$$\sum_{1 \leq i < j \leq n} P(A_i \cap A_j) = \binom{n}{2} \frac{1}{n(n-1)} = \frac{n(n-1)}{2!} \frac{1}{n(n-1)} = \frac{1}{2!}.$$

Similarly, for any $j \leq n$, $1, 2, \ldots, j$ are all fixed points with probability $\frac{(n-j)!}{n!}$. There are $\binom{n}{j}$ such sets, so the jth term of the sum will be:

$$\binom{n}{j} \frac{(n-j)!}{n!} = \frac{n!}{j!(n-j)!} \frac{(n-j)!}{n!} = \frac{1}{j!}.$$

We can conclude that:

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots + (-1)^{n+1} \frac{1}{n!},$$
$$P(\text{no fixed points}) = 1 - P(A_1 \cup A_2 \cup \cdots \cup A_n) = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \cdots + (-1)^n \frac{1}{n!}.$$

Recall also that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$, so:

$$P(\text{no fixed points}) \to e^{-1} \text{ as } n \to \infty.$$