1 Cumulative Distribution functions

Definition 1 Let ω be a continuous real-valued random variable. Then the cumulative distribution function of ω is:

$$F_\omega(x) = P(\omega \leq x).$$

Theorem 1 *Let ω be a continuous real-valued random variable with density function $f(\omega)$. Then the function defined by:*

$$F(x) = \int_{-\infty}^{x} f(\omega) d\omega$$

is the cumulative distribution function of ω. Furthermore,

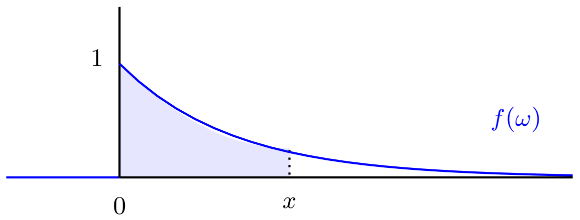
$$\frac{d}{dx} F(x) = f(x).$$

Proof we have $F(x) = P(\omega \leq x)$. Let $E = (-\infty, x]$. Then $P(E) = \int_{-\infty}^{x} f(\omega) d\omega$. The second part follows from the Fundamental Theorem of Calculus. $\square$

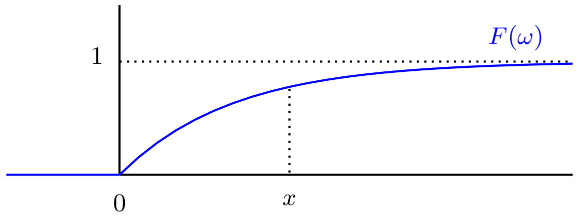
Example 1 Let the probability density be:

$$f(\omega) = \begin{cases} 0 & \text{if } \omega < 0 \\ e^{-\omega} & \text{if } \omega \geq 0 \end{cases},$$

then the graph of the density is:

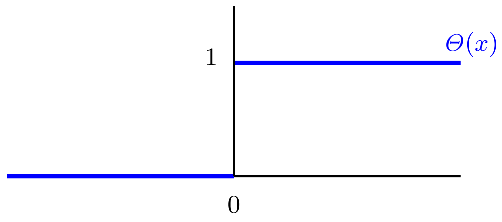


The shaded area is the cumulative distribution function at x. The graph of cumulative density is:



Example 2 A curious, but very important example is the *Dirac delta "function."* It's one of the most important mathematical objects in quantum mechanics. The reason I put *function* in quotation marks, is that it's actually a limit of functions. Here is an animation of functions $\delta_a(x) = \frac{1}{a\sqrt{\pi}} \mathrm{e}^{-x^2/a^2}$, then $\delta(x) = \lim_{a \to 0} \delta_a(x)$. The important thing about the delta is that it's 0 everywhere other than 0, and spikes up at 0 itself in a way that makes the integral 1. That means that it's not really a function itself, since it doesn't have a well defined value at 0. We can define it by:

$$\int_a^b \delta(x) = \begin{cases} 1 \text{ if } 0 \in [a, b] \\ 0 \text{ otherwise.} \end{cases}$$



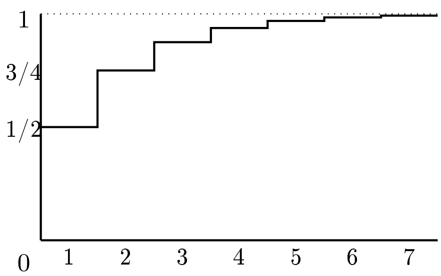
The cumulative distribution function for δ is much easier to plot. It's called the *Heaviside step function* $\Theta(x)$. This is what it looks like:

Example 3 Discrete cumulative distributions work the same way. Recall that if you toss a coin until the first time it comes up "heads," and set the random variable X to be the number of tosses, then:

$$P(X = n) = m(n) = \frac{1}{2^n}.$$

The cumulative distribution at n is again $P(X \leq n)$, which is:

$$P(X \leq n) = \sum_{i=1}^{n} m(i) = 1 - \frac{1}{2^n}.$$



2 Worksheet problems

1. Suppose you are watching a radioactive source that emits particles at a rate described by the exponential density

$$f(t) = \lambda e^{-\lambda t},$$

where $\lambda = 1$, so that the probability $P(0, T)$ that a particle will appear in the next T seconds is

$$P([0, T]) = \int_0^T \lambda e^{-\lambda t} dt.$$

Time right now is 0. Find the probability that a particle (not necessarily the first) will appear
- within the next second.
- within the next 3 seconds.
- between 3 and 4 seconds from now.
- after 4 seconds from now.

2* Take a stick of unit length and break it into three pieces, choosing the break points uniformly at random. What is the probability that the three pieces can be used to form a triangle?

3 Permutations

A *permutation* is a way of putting elements of a set in order. For example the set $\{a, b, c\}$ can be ordered in 6 ways: $abc, acb, bac, bca, cab, cba$. There are 3 ways of picking the first element and for each of those, two ways of ordering the other two.

When a set has n elements and we would like to order them all, there are $n!$ possibilities. The first element can be picked from all n, the second can be any one of them except for the one previously picked, which leaves $n-1$ options, etc. Therefore there are $n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$ options in total.

If we only want to pick some smaller number k out of n elements, in order, we only use the first k terms of $n!$. So there are

$$n \times (n-1) \times (n-2) \times \cdots \times (n-k+1) = \frac{n!}{k!}$$

possible ways to do it.

Example 4 Suppose that you encounter a combination lock with 10 digits, and know that the code is 4 digits long. There are $10^4 = 10,000$ possible combinations. Suppose that you use powder on the keys, and determine that 4 of them have a lot of grease and therefore are frequently used and you conclude that out that it's precisely these keys that are present in the code. Suddenly you only have $4! = 24$ possible combinations.

Example 5 (The Birthday Problem) There are 37 people in this class, and no one has a birthday on the 29th of February. What is the probability that two people have the same birthday?

There are 365^{37} possible ways in which people could have birthdays. How many are there such that no two poeple in the class have the same birthday? It's 365 options for the first person, 364 for the second and so on, all the way to 329. So there are $365!/328!$ ways in which no two people could share a birthday. But that's only about 15% of 365^{37}, so with about 85% probability there will be two people in the class who share a birthday.

3.1 Stirling's formula

Factorials are superexponential. The rate of growth of $n!$ is the same as $n^n e^{-n}\sqrt{2\pi n}$:

$$n! \sim n^n e^{-n}\sqrt{2\pi n}.$$