LECTURE 3 NOTES **Math 20 Fall 2014, Dartmouth College**

1 Infinities

The material in the "Infinities" section will not appear on exams for Math 20, just make sure that you know what "countable" means. But it is one of the most fascinating curiosities in mathematics.

1.1 Hilbert's paradox of the Grand Hotel

We all have some notions of what *infinity* is. Let me tell you a story. Once upon a time, there was an infinite hotel. the rooms were numbered, there was a room for every natural number and all rooms were occupied. And one day along came a new guest, and asked for a room. The receptionist was just about to tell him that they are fully booked, but the manager overheard it and suggested a solution: let's have every guest move to the NEXT room. So the guest in room 0 moved to room 1, the one in room 1 moved to room 2 and so on, and the new guest moved into room 0.

That story makes it clear that there are *countably* many rooms in the hotel, since they are numbered. You probably noticed that in the last definition of a probability distribution I used the word *countable.* If you've never come across it before, it means that there is a way to label all elements of a set with natural numbers such that there won't be any elements left. Alternatively, there's a way to put all elements in one room of the infinite hotel each. A set like that has exactly as many elements at the set of natural numbers. That's the smallest infinity.

1.2 Examples of sets that are countably infinite

Example 1 The set of integers $\mathbb{Z}$ is countable. We can assign rooms in the infinite hotel as follows:
For any integer n, n goes to room $2n$ is it's non-negative, and room $2(-n) - 1$ if it's negative.

Table 1 Room allocations for $\mathbb{Z}$.

room	0	1	2	3	4	5	6	7	8	...
occupant	0	-1	1	-2	2	-3	3	-4	4	...

There are many other possible assignments.

Example 2 The set $\mathbb{Z} \times \mathbb{Z}$, of pairs of integers: we can assign the elements to natural numbers in the order following the blue line in the picture.

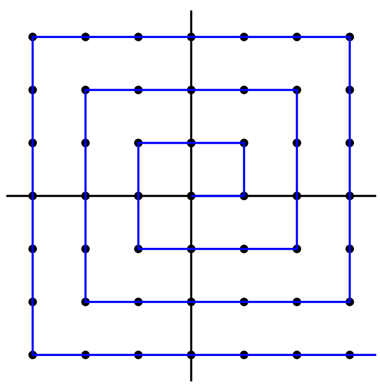


Table 2 Room allocations for $\mathbb{Z} \times \mathbb{Z}$.

room	0	1	2	3	4	5	6	7	8	...
occupant	(0,0)	(1,0)	(1,1)	(1,0)	(-1,-1)	(-1,0)	(-1,-1)	(0,-1)	(1,-1)	...

1.3 Diagonalization argument

Claim There are more numbers in the chunk $(0, 1)$ of the real number line than there are integers.

This means that if all of the numbers in $(0, 1)$ come to Hilbert's hotel, we won't be able to find them rooms. If we want to prove that something is impossible, we can use a *proof by contradiction*. It means that we show that what we want to show is impossible did in fact happen, it would lead to things that are illogical.

We start by assuming that it's possible to put all numbers in $(0, 1)$ separately in rooms of the infinite hotel, and show that however we do it, we can always find a number that doesn't have a room.

Proof To make it a little easier to think about, suppose that the rooms in the hotel are numbered $1, 2, 3, \ldots$, rather than starting from 0. The argument would work the same either way.

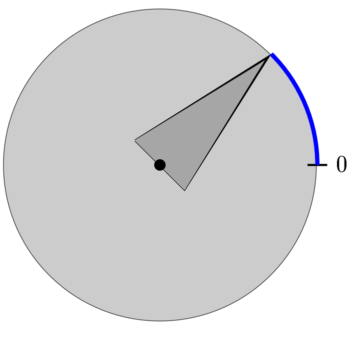
Suppose that there is some way of allocating real numbers in $(0, 1)$ to rooms in the Infinite Hotel, and number a_1 is taying in room 1, number a_2 is staying in room 2 and so on. Consider the decimal expansion of these numbers. Let the number a be the number between 0 and 1 such that in its nth decimal place is different from the number that is in nth decimal place of the number occupying nth room. For example, the first decimal place is the not the first decimal place of the number in room 1. The second decimal place is not the second decimal place of the number in room 2, and so on.

This number doesn't have a room! Suppose that this number is staying in room k. But its kth decimal place is different than the kth decimal place of the number in that room! We have arrived at a contradiction, so it's impossible to host all those numbers in the Infinite Hotel. □

1.4 Infinite binary tree

The set of finite binary sequences is countable, the set of infinite binary sequences is not. The above argument is very well explained on Wikipedia for the case of infinite binary sequences, under Cantor's diagonal argument.

2 Continuous Sample Spaces



Suppose that you have a spinner, that consists of a circle with unit circumference and a pointer. Spin the pointer, and let the random variable X be the length of the arc from 0, counterclockwise, to the tip of the pointer. The sample space is $\Omega = [0, 1)$.

We would like to construct a probability model in which every point on the circle is treated the same. then the probability that the tip of the pointer is in the upper half of the circle should be the same as in the lower half of the circle:

$$P(0 \leq X < 1/2) = P(1/2 \leq X < 1) = 1/2.$$

In general, for $[c, d) \subseteq [1, 0)$

$$P(X \in [c, d)]) = d - c.$$

Meanwhile, the probability that the pointer is at any specific point is 0. This makes intuitive sense since a point is infinitely small, but we can also prove it using limits.

What is the probability that $X = d$, for some $d \in [0, 1)$? Take any $c \in [0, 1)$ such that $c \leq d$?

$$\lim_{\epsilon \to 0} [P[]C \leq X < d + \epsilon] - P[c \leq d < d]] = \lim_{\epsilon \to 0} [d + \epsilon - c - (d - c)] = \lim_{\epsilon \to 0} [\epsilon] = 0.$$

That's why in the case of continuous probability, we talk about a *probability density* at a point, rather than a probability distribution.

Definition 1 The *probability density function* f is a function from Ω to $[0,\infty)$, such that for any subset E of Ω, the probability of E is:

$$P(E) = \int_E f(x)dx.$$

2.1 Uniform probability density

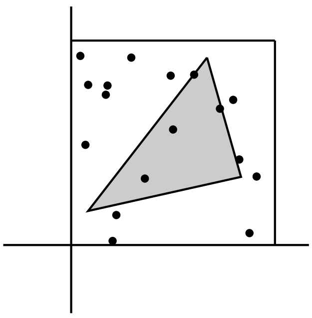
Based on the spinner argument, we need a density function $f(x)$ to be such that $\int_c^d f(x) = d - c$. This function is $f(x) = 1$.

2.2 Notation

We usually denote a continuous random variable with ω and an interval from x to y in Ω with (x, y). Then:

$$P(\omega \in (x,y)) = \int_x^y f(\omega)d\omega.$$

3 Monte Carlo Sampling



If we have a way of testing if a particular point is in a region, but don't know the area of the region we can use *Monte Carlo Sampling.* If we can enclose the region inside a rectangle, and pick a large number of points uniformly at random inside the rectangle, the proportion of the points inside the region will approximate the proportion of its area to the area of the rectangle.

Pseudocode:

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X=0
Do this n times:
    Draw a point uniformly at random inside the rectangle.
    If the point is in the shaded region, add 1 to X.
Area = Area of the rectangle * X/n
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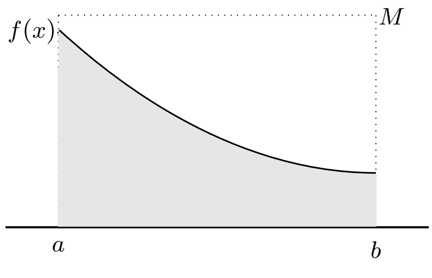
For some functions, it is very hard to compute their integrals. In such cases, we can use Monte Carlo Sampling to estimate their integral numerically in the following way.

If we wish to estimate $\int_a^b f(x)dx$, and $0 \le f(x) \le M$ for some number M for all x in $[a, b]$, we can enclose the function in a rectangle. In order to test if a point (x, y) is in the shaded region, we need only test if $y \le f(x)$.

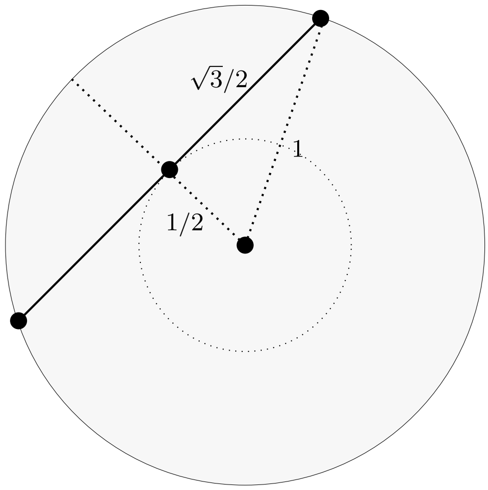


4 Bertrand's Paradox

A *chord* of a circle is a line segment such that both endpoints belong to the circumference of the circle. Suppose that we want to draw the chord uniformly at random, on a ricle with unit radius. What is the probability that the length of the chord is more than $\sqrt{3}$?

It all depends on how we draw the chord. Here are some of the ways we could do it:

1. Draw the endpoints of the chord uniformly at random.
2. Any point inside the cicrle can serve as the midpoint of a chord. Draw a point in the circle uniformly at random.
   a) Using polar coordinates (r, θ).
   b) Using cartexian coordinates (x, y).



The conditions for the chord to be at least $\sqrt{3}$ long, are as follows:

1. Once we pick the first endpoint, the chord will have the required length if the other endpoint belongs to the 1/3 of the circumference that is far enough from the first one. The probability of that is 1/3.
2. a) If the midpoint has $r \leq 1/2$, the chord will have the required length. The probability of that is 1/2.
   b) If the midpoint is in the circle of radius 1/2, co-centric with the original circle, the chord has required length. the probability of that is 1/4.

Everything depends on how we set up the experiment. In a sense, on what we mean by “random."