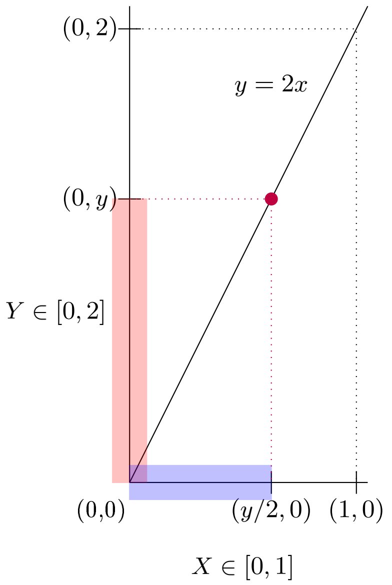
Lecture 10: Worksheet explained **Math 20 Fall 2014, Dartmouth College**

1 Functions of a Random Variable: Cumulative Probability Distribution

Consider a continuous random variable X with sample space $\Omega = [0, 1]$ and cumulative distribution function:

$$F_X(x) = P[X \leq x].$$

Let $y = 2x$ be a function of X. If $Y = 2X$ is another random variable, what is the sample space of Y? What is the cumulative distribution function $F_Y(y) = P[Y \leq y]$ in terms of F_X and y?



The sample space of Y is $[0, 2]$. The value of Y is a random variable, because it's generated from the random input X. Now, the cumulative distribution function $F_Y(y)$ is the probability that Y falls between 0 and the value y. (Remember, y is a number, Y is the whole random variable.)

The probability that Y falls between 0 and y is the same as **the probability that X is such that Y falls between 0 and y.** And we know the probability density of X.

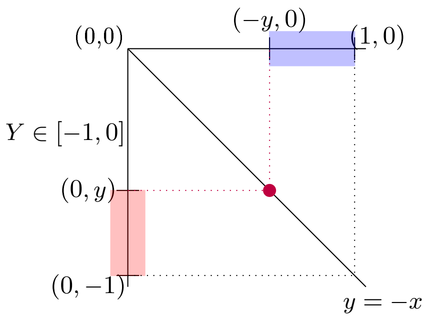
$$Y \in [0, y] \text{ IF AND ONLY IF } X \in [0, y/2].$$

This is illustrated on the left. The probability that X falls between 0 and $y/2$ is, by definition, $F_X(y/2)$. So:

$$F_Y(y) = F_X(y/2).$$

In general, for a strictly increasing function $Y = g(X)$, Y falls between 0 and y if and only if X falls between 0 and $g^{-1}(y) = x$, i.e. the value x such that $g(x) = y$. So the cumulative distribution functions are related by:

$$F_Y(y) = F_X(g^{-1}(y)).$$



What would be the version of the above argument for $Y = -X$? It's very similar, but we need to remember that the smaller the Y, the higher the X. So if we want to find $F_Y(y) = P[Y \leq y]$, we need to first find x such that $g(x) = y$, in this case $x = -y$, and then take $P[X \geq x]$. This is $1 - P[x \leq x] = 1 - F_X(x)$. Putting it all together, we get:

$$F_Y(y) = 1 - F_X(-y).$$

In general, for a strictly decreasing function $Y = g(X)$:

$$F_Y(y) = 1 - F_X(g^{-1}(x)).$$

2 Functions of a Random Variable: Probability Density

If $g(X)$ is strictly increasing and $Y = g(X)$, then the densities f_X of X and f_Y of Y are related by:

$$f_Y(y) = f_X(g^{-1}(y)) \frac{d}{dy} g^{-1}(y)$$

If $g(X)$ is strictly decreasing and $Y = g(X)$, then the densities f_X of X and f_Y of Y are related by:

$$f_Y(y) = -f_X(g^{-1}(y)) \frac{d}{dy} g^{-1}(y)$$

What if $f_X(g^{-1}(y))$ for any y if X takes values in $[0, 1]$ uniformly at random?

Let the random variable U be chosen from the interval $[0,1]$ with uniform density. Find the density functions $f_Y(y)$ for the random variables:

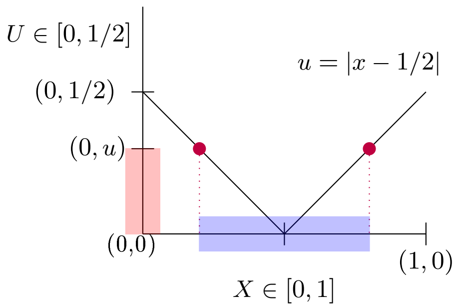
a. $Y = U + 2$
b. $Y = U^3$
c. $Y = 1/(U+1)$
d. $Y = \log(U+1)$

Notice that since $f_U(u) = 1$ for all values $U = u$, for any y we have $f_U(g^{-1}(y)) = 1$. So the expressions from the previous page in this case are:

$$f_Y(y) = \frac{d}{dy} g^{-1}(y) \text{ for increasing functions and } f_Y(y) = -\frac{d}{dy} g^{-1}(y) \text{ for decreasing ones.}$$

The answers are at the bottom of the page.

What is the cumulative probability distribution $F_U(u)$, if X is taken from $[0,1]$ uniformly at random, and $U = |X - 1/2|$?



We need $P[U \le u]$. So we want all values x such that $|x - 1/2| \le 1/2$. This interval will be twice the length of $[0, u]$ (see picture.) In general, if we are dealing with a function that changes direction, we might need to deal with it piece by piece.

If we're instead looking for the probability density, f_U, then again the f_X term will be 1. However, we need the $f_U(u) = -\frac{d}{du} g^{-1}(u)$ formula wherever the function is decreasing, and the $f_U(u) = \frac{d}{du} g^{-1}(u)$ one where it's increasing. Fortunately, in this case the minus cancels out with the derivative (try it!)

3 Exponential density

The exponential density function is:

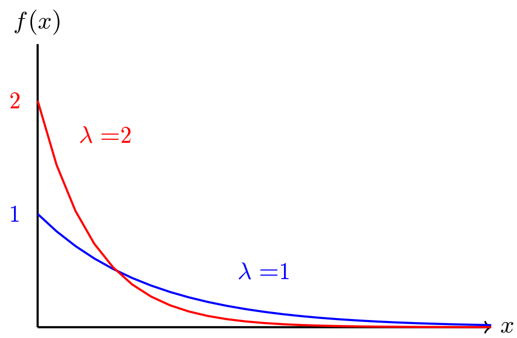
$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } 0 \le x < \infty \\ 0, & \text{otherwise} \end{cases}$$

for some positive constant λ.

What is the cumulative distribution function $F(x)$ in this case?

$$F(x) = \int_0^x f(t)dt = \int_0^x \lambda e^{-\lambda t} dt = \left[\frac{\lambda e^{-\lambda t}}{-\lambda}\right]_0^x = \left[-e^{-\lambda t}\right]_0^x = -e^{-\lambda x} + 1$$

$$F(x) = 1 - e^{-\lambda x}$$



Show that for any positive constants r and s, $P[X > r + s | X > r] = P[X > s]$. What does that mean, in words?

Notice that the intersection of $X > r$ and $X > r + s$ is simply $X > r + s$.

$$LHS = P[X > r + s | X > r] = \frac{P[X > r + s]}{P[X > r]} = \frac{1 - F(r+s)}{1 - F(r)} = \frac{e^{\lambda(r+s)}}{e^{\lambda r}} = e^{\lambda s} = 1 - F(s) = P[X > s] = RHS$$

This means that no matter where you are in $[0, \infty)$, if X hasn't happened yet, then the probability that X happens in the next period s is the same. For example, if someone is talking on the phone then no matter how long they've been talking the probability that they hang up in the next minute is the same. (Yes, length of phone calls obeys this density.)

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a. $f_Y = 1$
b. $f_Y = \frac{d}{dy}(y^{1/3}) = \frac{u^{-2/3}}{3}$
c. $f_Y = -\frac{d}{dy}(\frac{1}{y} - 1) = -\frac{-1}{y^2} = \frac{1}{y^2}$
d. $f_Y = \frac{d}{dy}(e^y - 1) = e^y$