LECTURE 1 SUMMARY

1 Probability and Randomness

If we toss a fair coin, it has “probability" 1/2 of coming up heads, and “probability" 1/2 of coming up tails. This means that *if we toss the coin many times, it will come up heads about half the time.* In fact, we can be pretty sure that the more times we toss the coin, the closer to 1/2 the proportion of times it comes up heads will get.

Notation:

Tossing a coin is an *experiment* that can be repeated, and heads (H) and tails (T) are two possible outcomes.

$P(H) = 1/2$ is the probability of outcome “heads."
$P(T) = 1/2$ is the probability of outcome “tails."

Outcomes H and T can never occur at the same time, they are *disjoint* or *mutually exclusive.*

$P(H) + P(T) = 1$

2 Rolling two dice

Suppose we roll two six-sided dice. What is a good bet on the sum of the outcomes on the two dice?

7 is the sum more often than 3. But ⚀ ⚅ comes oup with the same probability as ⚀ ⚁. What’s different?

Table 1 Sum of the outcomes on the two dice.

	⚀	⚁	⚂	⚃	⚄	⚅
⚀	2	3	4	5	6	7
⚁	3	4	5	6	7	8
⚂	4	5	6	7	8	9
⚃	5	6	7	8	9	10
⚄	6	7	8	9	10	11
⚅	7	8	9	10	11	12

There are more ways in which 7 can come up that 3.

Question: Are there really more ways in which 7 can come up than 6? To make 6, we need 1+5, 2+4, or 3+3. To make 7, we need 1+6, 2+5, or 3+4. What’s going on here? Try to figure it out.

3 Set operations

Suppose that one die is red and the other one blue. Let the event A be “the red die comes up as ⚀" and the event B be “the sum is 7." These are not disjoint events, it is possible that both statements are true. For the following, refer to illustrations on page 22 of the textbook.

$P(A) = 1/6$
$P(B) = 1/6$

Intersection: both statements are true. $P(A \cap B) = 1/36$ is the probability that red die comes up ⚀ and the sum is 7.

Union: at least one of the statements is true.
$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 11/36$ is the probability that red die comes up ⚀, the sum is 7 or both.

Compliment: the statement is not true. $P(\bar{B}) = 1 - P(B) = 5/6$ is the probability that the sum is not 7.

Set difference: statement A is true, but statement B is not.

$P(A - B) = P(A) - P(A \cap B) = 1/6 - 1/36 = 5/36$ is the probability that the red die comes up ⚀ but the sum is not 7.

Definition 1 The outcome of a randomized experiment is a *random variable*, often denoted with letter X. The set of all possible outcomes Ω is called the *sample space*. If the sample space is finite or countable, then the variable X is *discrete*.

Example 1 When we toss a coin, $\Omega = \{H, T\}$. Then either $X = H$ or $X = T$.

Example 2 When we roll two dice, red and blue, and write the result on the red die first:

$\Omega = \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),$
$(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),$
$(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),$
$(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),$
$(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),$
$(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}.$

Example 3 When we roll two dice and record the sum of the two, $\Omega = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$.

Definition 2 An *event* can be any statement about the outcome, alternatively, we can view it as a subset of the sample space.

In the setup of Example 2, "the red die comes up 1 and the sum is 7" is an event. call it event A. Then $P(A) = P(X = (1,6))$, the "probability that X is $(1,6)$." If event B is "the red die comes up odd and the sum is 7", then:

$P(B) = P(X = (1,6)) + P(X = (3,4)) + P(X = (5,2)).$